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Invariant Finite Element Model for Composite Structures: The Generalized Unified Formulation

Luciano Demasi*
San Diego State University, San Diego, California 92182

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The generalized unified formulation presented here is a modern approach for the finite element analysis of sandwich plates and in general multilayered structures. A large variety of types of theories with any combination of orders of expansion for the different displacement variables can be obtained from the expansion of six 1 × 1 arrays (the kernels or fundamental nuclei of the generalized unified formulation). Each of the displacement variables is independently treated and different orders of expansions for the different unknowns can be chosen. Because infinite combinations can be freely chosen for the displacements, the generalized unified formulation allows the user to write ∞^3 higher-order shear deformation theories, ∞^3 zig–zag theories, and ∞^3 layerwise theories with a single invariant formulation implemented in a single finite element method code. The six independent fundamental nuclei are formally invariant with respect to the order used for the expansion or with respect to the type of theory. The generalized unified formulation can be also adopted for mixed variational statements. In such cases the number of independent kernels would be different but their size would still be 1 × 1 arrays. This paper assesses bending of sandwich structures. Analytical and elasticity solution are compared. The effect of the zig–zag form of the displacement is discussed.

Nomenclature

${}^{t}D^{\alpha_{u_x}\beta_{u_x}}_{u_x u_x}$	=	invariant pressure kernel (1 × 1 array)
${}^{t}D^{\alpha_{u_y}\beta_{u_y}}_{u_y u_y}$	=	invariant pressure kernel (1 × 1 array)
${}^{t}D^{\alpha_{u_z}\beta_{u_z}}_{u_z u_z}$	=	invariant pressure kernel (1 × 1 array)
${}^{b}D^{\alpha_{u_x}\beta_{u_x}}_{u_x u_x}$	=	invariant pressure kernel (1 × 1 array)
${}^{b}D^{\alpha_{u_y}\beta_{u_y}}_{u_y u_y}$	=	invariant pressure kernel (1 × 1 array)
${}^{b}D^{\alpha_{u_z}\beta_{u_z}}_{u_z u_z}$	=	invariant pressure kernel (1 × 1 array)
$\boldsymbol{\varepsilon}_n$	=	vector containing the out-of-plane strains
$\boldsymbol{\varepsilon}_p$	=	vector containing the in-plane strains
ε_{xx}, ε_{yy}, γ_{xy}	=	in-plane strains
ε_{zz}, γ_{xz}, γ_{yz}	=	out-of-plane strains
$K^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x}$	=	invariant kernel (1 × 1 array)
$K^{k\alpha_{u_x}\beta_{u_y}}_{u_x u_y}$	=	invariant kernel (1 × 1 array)
$K^{k\alpha_{u_x}\beta_{u_z}}_{u_x u_z}$	=	invariant kernel (1 × 1 array)
$K^{k\alpha_{u_y}\beta_{u_y}}_{u_y u_y}$	=	invariant kernel (1 × 1 array)
$K^{k\alpha_{u_y}\beta_{u_z}}_{u_y u_z}$	=	invariant kernel (1 × 1 array)
$K^{k\alpha_{u_z}\beta_{u_z}}_{u_z u_z}$	=	invariant kernel (1 × 1 array)
m, n	=	wave numbers
N_{u_x}	=	order of expansion used for the displacement u_x
N_{u_y}	=	order of expansion used for the displacement u_y
N_{u_z}	=	order of expansion used for the displacement u_z
u_x	=	displacement in the x direction
u_y	=	displacement in the y direction
u_z	=	displacement in the z direction
$\hat{u}_x$, $\hat{u}_y$	=	dimensionless in-plane displacements
$\hat{u}_z$	=	dimensionless out-of-plane displacement
x, y	=	in-plane coordinates
${}^{x}F_t$, ${}^{y}F_t$, ${}^{z}F_t$	=	known functions used in the expansion along the thickness
${}^{x}F_b$, ${}^{y}F_b$, ${}^{z}F_b$	=	known functions used in the expansion along the thickness
${}^{x}F_l$, ${}^{y}F_m$, ${}^{z}F_n$	=	known functions used in the expansion along the thickness
z	=	out-of-plane coordinate
z_{bot_k}	=	z coordinate of the bottom surface of layer k
z_{top_k}	=	z coordinate of the top surface of layer k
α	=	thickness primary master index
β	=	thickness secondary master index
ζ_k	=	nondimensional coordinate [see Eq. (39)]
$\boldsymbol{\sigma}_p$	=	vector containing the in-plane stresses
$\boldsymbol{\sigma}_n$	=	vector containing the out-of-plane stresses
$\hat{\sigma}_{zx}$, $\hat{\sigma}_{zy}$, $\hat{\sigma}_{zz}$	=	dimensionless out-of-plane stresses
$\hat{\sigma}_{xx}$, $\hat{\sigma}_{yy}$, $\hat{\sigma}_{xy}$	=	dimensionless in-plane stresses
σ_{xx}, σ_{yy}, σ_{xy}	=	in-plane stresses
σ_{zz}, σ_{xz}, σ_{yz}	=	out-of-plane stresses

Subscripts

G	=	quantity calculated by using the geometric relations [see Eq. (11)]
H	=	quantity calculated by using Hooke's law
k	=	referred to layer k
u_x	=	referred to displacement u_x
u_y	=	referred to displacement u_y
u_z	=	referred to displacement u_z

Superscripts

k	=	referred to layer k
T	=	transpose

Introduction

Background and Motivation

FINITE element method (FEM) [1–3] is a powerful tool widely used in industry. With the growing computational capabilities of modern computers, increasingly complex problems can be solved.

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*Assistant Professor, Department of Aerospace Engineering and Engineering Mechanics; ldemasi@mail.sdsu.edu. Member AIAA.

Most aerospace structures can be modeled with shell and plate models. Therefore, the importance of finite element applications of such structures is great. Different problems have been solved in the past: free vibration of plates [4–11], analysis of sandwich structures [12–16], thermoelastic problems [17], piezoelectric plates and multifield loadings [18–22].

With focus on axiomatic models (alternative approaches based on asymptotic formulations are possible [23–26]), each existing method presents a range of applicability and when the hypotheses used to formulate the theory are no longer satisfied the approach has to be replaced with another one usually named by the authors as refined theory or improved theory. In the framework of the mechanical case the classical plate theory, also known as Kirchoff theory [27], has the advantage of being simple and reliable for thin plates. However, if there is strong anisotropy of the mechanic properties, or if the composite plate is relatively thick, other advanced models such as first-order shear deformation theory (FSDT) are required [28–30]. Higher-order shear deformation theories (HSDT) have also been used [7–9], giving the possibility to increase the accuracy of numerical evaluations for moderately thick plates. But even these theories are not sufficient if local effects are important or accuracy in the calculation of transverse stresses is sought. Therefore, more advanced plate theories have been developed to include zig–zag effects [31–45].

When a new engineering problem with unknown behavior is analyzed, the user may attack it with the most simple formulation or with the most advanced and refined models or with an intermediate solution which is a compromise between the accuracy and the CPU time needs. This operation is usually costly. For example, the different capabilities required for this approach are not available. In that case in-house models are often created. But these in-house approaches obviously do not contain all possible models between the least, and usually fast, accurate one and the most advanced, and usually time consuming, one. Even if different approaches are considered, each one requires a different formulation of the FE stiffness matrix. A major problem may arise in that case. The implemented capability may not be sufficient to solve the challenging case the user is facing or the CPU time is unacceptable and a less accurate theory would be sufficient. These aspects are also problem-dependent and even if the solution of a previous case was satisfactory, the same method may no longer be adequate for a new task.

The problem of adequately modeling the behavior of the real structure is then very challenging. For example, if two-dimensional multilayered structures cases are considered, it may be necessary to abandon the so-called equivalent single layer models and adopt layerwise models [10,46–53] in which the variables are layer-dependent. If the anisotropy in the thickness direction is very strong, the accuracy can be preserved by increasing the number of layers used in the mathematical model or/and by increasing the orders of the axiomatic expansion along the thickness for the different variables. If this approach is chosen, the use of solid elements can be avoided and the advantage of having a two-dimensional mesh can be preserved. Versatility is an important parameter in the choice of an approach among available alternative methods. An optimization problem or a probabilistic analysis (Monte–Carlo) needs to be very fast, whereas a detailed study of the interlaminar stresses requires an advanced model. The multitheory and multifidelity architecture presented in this paper meets all the above mentioned requirements and is a computationally efficient technique for the modelization of most of the structures of engineering interest. Piezoelectric layers and functionally graded materials can be considered as well. Multifidelity models and mesh dependency reduction in shape optimization software can be achieved by using the present approach.

Contributions of this Work

An invariant model, named generalized unified formulation (GUF) is introduced. It can be considered a generalization of the unified formulation [54] (named here as Carrera's unified formulation). With the present GUF all the needs earlier mentioned are satisfied. In particular, the GUF allows to treat each displacement variable independently from the others and this could lead to the derivation of new FEM numerical techniques. (If a mixed variational statement is used or if the case includes multifield quantities, other variables may be modeled.) The orders of each variable are freely chosen by the user and can be different than the orders used for the thickness expansion of the other variables. This property means that practically all classical equivalent single layer models can be addressed by the GUF. For example, an advanced higher-order shear deformation theory in which the in-plane displacements are expanded along the thickness by using a cubic polynomial and in which the out-of-plane displacement is parabolic can be represented with GUF as well as the theory in which the orders for the in-plane displacements are parabolic and the out-of-plane displacement is linear. Any combination is allowed. But what is important is that all the possible combinations that can be generated (which are ∞^3 since three displacement variables u_x, u_y and u_z are here considered) are obtained from the expansion of six kernels which are invariant with respect to the order adopted for the displacements variables. That is, all the higher-order shear deformation theories with any combination of orders can be generated from the same six invariant kernels. The kernels of the GUF are 1×1 matrices. If other variational statements different than the principle of virtual work are used, the number of independent kernels and their mathematical form will be different but the size will always be 1×1.

It is possible to demonstrate that when the so-called zig–zag form of the displacements is included by adopting Murakami's zig–zag function [31], the same invariant kernels can describe the ∞^3 zig–zag theories that could be generated by changing the orders of the expansions of the displacements. But these features are not limited to the equivalent single layer models. Layerwise theories with any combination of orders can be obtained, again, from the same six invariant 1×1 kernels. The GUF clearly allows to have all these infinite models in the same software. From the least accurate to the most advanced one. The freedom to change the orders of expansion in a layerwise model allows to reach a quasi-3D solution without the actual need of solid elements and so with the preservation of the advantages of having a two-dimensional mesh. This is very useful when local effects are important or when the analyzed structure may not be considered as a "thin structure" and so when the classical two-dimensional formulation fails.

Another advantage of the GUF is in the possibility of a numerical experiment or a sensitivity analysis of the problem under investigation. For example, it is possible to explore if a particular configuration of the constraints and loads is affected more by the order of expansion of one variable with respect to the orders used for the other variables. This feature is particularly appealing in the mixed or multifield cases in which the order of expansion of some unknown quanties may be determinant to have the correct numerical simulation. But this is not all. The GUF can be used to "adapt" the software to the optimization or probabilistic simulation. The user can in fact study with a small number of runs the best combination of type of theory (e.g., equivalent single layer theory with zig–zag function vs a higher-order shear deformation theory) and orders that minimize the CPU time and maximize the accuracy within the requested level. Then the user can start the optimization or probabilistic study knowing that the software gives the best performances because it is tailored to the case under investigation. The GUF could also be used to design automatic and intelligent FEM codes: the codes could understand (with a designed algorithm) where the accuracy is required the most and so they could try any of the types of theories and any of the ∞^3 combinations allowed by GUF and improve the accuracy of the prediction. As stated earlier this capability is not limited to the pure mechanical case presented in this paper and it can be easily extended to the multifield problems and functionally graded materials.

A final but not less important advantage of the GUF is in the educational possibilities for young engineers. The engineers could learn a single formulation (GUF) and apply all the powerful features of GUF to a very vast class of problems without the need of learning new formulations of commercial software. With the GUF the classical axiomatic approaches and the distinction between theories are just an exercise and the user can move from one approach to another one without difficulties. The GUF can be then considered a modern view of multilayered plates and shells with possible applications in high fidelity codes in the different disciplines such as aeroelasticity.

Classification of the Theories

The main feature of the GUF is that the descriptions of layerwise theories, higher-order shear deformation theories, and zig–zag theories of any combination of orders do not show any formal differences and can all be obtained from six invariant kernels. So, with just one theoretical model an infinite number of different approaches can be considered. For example, in the case of moderately thick plates a higher-order theory could be sufficient, but for thick plates layerwise models may be required. With GUF the two approaches are formally identical because the kernels are invariant with respect to the type of theory.

In the present work the concepts of type of theory and class of theories are introduced. The following types of displacement-based theories are discussed. The first type is named as advanced higher-order shear deformation theories (AHSDT). These theories are equivalent single layer models because the displacement field is unique and independent of the number of layers. The effects of the transverse normal strain ε_{zz} are retained.

The second type of theories is named as advanced zig–zag theories (AZZT). These theories are equivalent single layer models and the so-called zig–zag form of the displacements is taken into account by using Murakami's zig–zag function (MZZF). The effects of the transverse normal strain ε_{zz} are included. The third type of theories is named advanced layerwise theories (ALWT). These theories are the most accurate ones because all the displacements have a layerwise description. The effects of the transverse normal strain ε_{zz} are included as well. These models are necessary when local effects need to be described. The price is of course (in FEM applications) in higher computational time. An infinite number of theories which have a particular logic in the selection of the used orders of expansion is defined as class of theories. For example, the infinite layerwise theories which have the displacements u_x, u_y and u_z expanded along the thickness with a polynomial of order N are a class of theories. The infinite theories which have the in-plane displacements u_x and u_y expanded along the thickness with order N, the out of plane displacement expanded along the thickness with order $N-1$ are another class of theories.

GUF for Multilayered Composite Plates

Both layerwise and equivalent single layer models are axiomatic approaches. That is, the unknowns are expanded along the thickness by using a chosen series of functions.

When the principal of virtual displacements is used, the unknowns are the displacements u_x, u_y, and u_z. When other variational statements are used the unknowns may also be all or some of the stresses and other quantities as well (multifield case).

The GUF is introduced here considering a generic layer k of a multilayered plate structure. This is the most general approach and the equivalent single layer theories, which consider the displacement unknowns to be layer-independent, can be derived from this formulation with some simple formal techniques as will be demonstrated in this paper. Consider a theory denoted as Theory I, in which the displacement in x direction u_x^k has 4 degrees of freedom. Here, by degrees of freedom it is intended the number of unknown quantities that are used to expand a variable. In the case under examination 4 degrees of freedom for the displacement u_x^k means that four unknowns are considered. Each unknown multiplies a known function of the thickness coordinate z. Where the origin of the coordinate z is measured is not important. However, from a practical point of view it is convenient to assume that the middle plane of the plate is also the plane with $z=0$. This assumption does not imply that there is a symmetry with respect to the plane $z=0$. The formulation is general.

For layer k the following relation holds: $z_{\text{bot}_k} \le z \le z_{\text{top}_k}$. z_{bot_k} is the global coordinate z of the bottom surface of layer k and z_{top_k} is the global coordinate z of the top surface of layer k (see Fig. 1). $h_k = z_{\text{top}_k} - z_{\text{bot}_k}$ is the thickness of layer k and h is the thickness of the plate.

In the case of theory I, u_x^k is expressed as follows:

$$u_x^k(x,y,z) = \overbrace{f_1^k(z)}^{\text{known}} \cdot \overbrace{u_{x_1}^k(x,y)}^{\text{unknown\#1}} + \overbrace{f_2^k(z)}^{\text{known}} \cdot \overbrace{u_{x_2}^k(x,y)}^{\text{unknown\#2}} + \underbrace{f_3^k(z)}_{\text{known}} \cdot \underbrace{u_{x_3}^k(x,y)}_{\text{unknown\#3}} + \underbrace{f_4^k(z)}_{\text{known}} \cdot \underbrace{u_{x_4}^k(x,y)}_{\text{unknown\#4}} \qquad z_{\text{bot}_k} \le z \le z_{\text{top}_k} \quad (1)$$

The functions $f_1^k(z)$, $f_2^k(z)$, $f_3^k(z)$ and $f_4^k(z)$ are known functions (axiomatic approach). These functions could be, for example, a series of trigonometric functions of the thickness coordinate z. Polynomials (or even better orthogonal polynomials) could be selected. In the most general case each layer has different functions. For example, $f_1^k(z) \ne f_1^{k+1}(z)$. The next formal step is to modify the notation.

The following functions are defined:

$$\begin{aligned} {}^xF_t^k(z) &= f_1^k(z) \qquad {}^xF_2^k(z) = f_2^k(z) \\ {}^xF_3^k(z) &= f_3^k(z) \qquad {}^xF_b^k(z) = f_4^k(z) \end{aligned} \quad (2)$$

The logic behind these definitions is the following. The first function $f_1^k(z)$ is defined as ${}^xF_t^k$. Notice the superscript x. It was added to clarify that the displacement in x direction, u_x^k, is under investigation. The subscript t identifies the quantities at the top of the plate and, therefore, will be useful in the assembling of the stiffness matrices in the thickness direction.

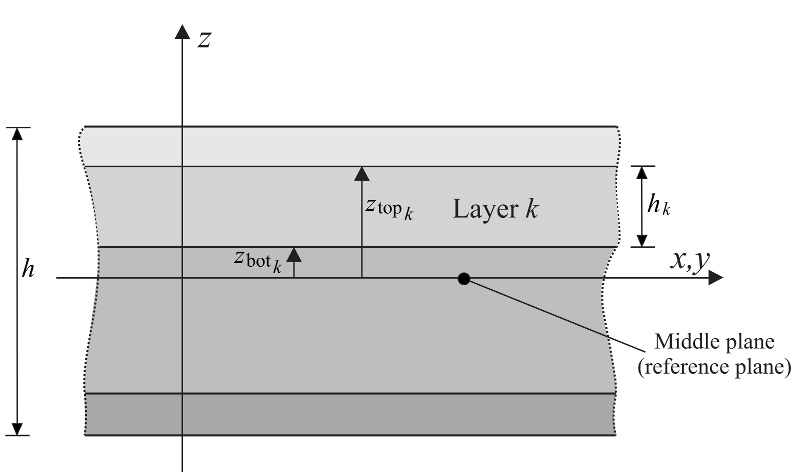


Fig. 1 Multilayered plate: notations and definitions.

The last function $f_4^k(z)$ is defined as ${}^xF_b^k$. Notice again the superscript x. The subscript b means bottom and, again, its utility will be clear when the matrices are assembled.

The intermediate functions $f_2^k(z)$ and $f_3^k(z)$ are defined simply as ${}^xF_2^k$ and ${}^xF_3^k$. To be consistent with the definitions of Eq. (2), the following unknown quantities are defined:

$$u_{x_t}^k(x,y) = u_{x_1}^k(x,y) \qquad u_{x_b}^k(x,y) = u_{x_4}^k(x,y) \tag{3}$$

Using the definitions reported in Eqs. (2) and (3), Eq. (1) can be rewritten as

$$u_x^k(x,y,z) = \overbrace{{}^xF_t^k(z)}^{\text{known}} \cdot \overbrace{u_{x_t}^k(x,y)}^{\text{unknown\#1}} + \overbrace{{}^xF_2^k(z)}^{\text{known}} \cdot \overbrace{u_{x_2}^k(x,y)}^{\text{unknown\#2}} + \underbrace{{}^xF_3^k(z)}_{\text{known}} \cdot \underbrace{u_{x_3}^k(x,y)}_{\text{unknown\#3}} + \underbrace{{}^xF_b^k(z)}_{\text{known}} \cdot \underbrace{u_{x_b}^k(x,y)}_{\text{unknown\#4}} \qquad z_{\text{bot}_k} \le z \le z_{\text{top}_k} \tag{4}$$

It is supposed that each function of z is a polynomial. The order of the expansion is then 3 and indicated as $N_{u_x}^k$. A different order of expansion may be selected for each layer. Thus, in general $N_{u_x}^k \neq N_{u_x}^{k+1}$. If the functions of z are not polynomials (for example, this is the case if trigonometric functions are used) then $N_{u_x}^k$ is just a parameter related to the number of terms or degrees of freedom used to describe the displacement u_x^k in the thickness direction. The expression representing the displacement u_x^k (see Eq. (4)) can be put in a compact form typical of the GUF presented here. In particular it is possible to write:

$$u_x^k(x,y,z) = {}^xF_{\alpha_{u_x}}^k(z) \cdot u_{x\alpha_{u_x}}^k(x,y)$$
$$\alpha_{u_x} = t, l, b; \qquad l = 2, \ldots, N_{u_x}^k \tag{5}$$

where, in the example, $N_{u_x}^k = 3$. The thickness primary master index α has the subscript u_x. This subscript from now on will be called *slave index*. It is introduced to show that the displacement u_x is considered. Figure 2 explains these definitions. Consider another example. Suppose that the displacement u_x^k of a particular theory is expressed with 3 degrees of freedom. In that case it is possible to write:

$$u_x^k(x,y,z) = \overbrace{f_1^k(z)}^{\text{known}} \cdot \overbrace{u_{x_1}^k(x,y)}^{\text{unknown\#1}} + \overbrace{f_2^k(z)}^{\text{known}} \cdot \overbrace{u_{x_2}^k(x,y)}^{\text{unknown\#2}} + \overbrace{f_3^k(z)}^{\text{known}} \cdot \overbrace{u_{x_3}^k(x,y)}^{\text{unknown\#3}} \tag{6}$$

By adopting the definitions earlier used for the case of 4 degrees of freedom it is possible to rewrite Eq. (6) in the following equivalent form:

$$u_x^k(x,y,z) = \overbrace{{}^xF_t^k(z)}^{\text{known}} \cdot \overbrace{u_{x_t}^k(x,y)}^{\text{unknown\#1}} + \overbrace{F_2^k(z)}^{\text{known}} \cdot \overbrace{u_{x_2}^k(x,y)}^{\text{unknown\#2}} + \overbrace{{}^xF_b^k(z)}^{\text{known}} \cdot \overbrace{u_{x_b}^k(x,y)}^{\text{unknown\#3}} \tag{7}$$

which can be put again in the form shown in Eq. (5) with $N_{u_x}^k = 2$. In general $N_{u_x}^k = \text{DOF}_{u_x}^k - 1$, where $\text{DOF}_{u_x}^k$ is the number of degrees of freedom (at layer level) used for the displacement u_x^k. In the case of zig–zag theories it is possible to demonstrate that $N_{u_x}^k = \text{DOF}_{u_x}^k - 2$ because 1 degree of freedom is used for the zig–zag function.

The minimum number of degrees of freedom is chosen to be 2. This is a choice used to facilitate the assembling in the thickness direction. In fact, the top and bottom terms will be always present. In the case in which $\text{DOF}_{u_x}^k = 2$ the GUF is simply

$$u_x^k(x,y,z) = {}^xF_{\alpha_{u_x}}^k(z) \cdot u_{x\alpha_{u_x}}^k(x,y) \qquad \alpha_{u_x} = t, b \tag{8}$$

In this particular case the l term of Eq. (5) is not present.

An infinite number of theories can be included in Eq. (5). It is in fact sufficient to change the value of $N_{u_x}^k$. It should be observed that formally there is no difference between two distinct theories (obtained by changing $N_{u_x}^k$). It is deduced that ∞^1 theories can be represented by Eq. (5).

The other displacements u_y^k and u_z^k can be treated in a similar fashion. The GUF for all the displacements is the following:

$$u_x^k = {}^xF_t u_{x_t}^k + {}^xF_l u_{x_l}^k + {}^xF_b u_{x_b}^k = {}^xF_{\alpha_{u_x}} u_{x\alpha_{u_x}}^k$$
$$\alpha_{u_x} = t, l, b; \qquad l = 2, \ldots, N_{u_x}$$
$$u_y^k = {}^yF_t u_{y_t}^k + {}^yF_m u_{y_m}^k + {}^yF_b u_{y_b}^k = {}^yF_{\alpha_{u_y}} u_{y\alpha_{u_y}}^k$$
$$\alpha_{u_y} = t, m, b; \qquad m = 2, \ldots, N_{u_y}$$
$$u_z^k = {}^zF_t u_{z_t}^k + {}^zF_n u_{z_n}^k + {}^zF_b u_{z_b}^k = {}^zF_{\alpha_{u_z}} u_{z\alpha_{u_z}}^k$$
$$\alpha_{u_z} = t, n, b; \qquad n = 2, \ldots, N_{u_z} \tag{9}$$

In Eq. (9), for simplicity it is assumed that the type of functions is the same for each layer and that the same number of terms is used for each layer. This assumption will make it possible to adopt the same GUF for all types of theories, and layerwise and equivalent single layer theories will not show formal differences. This concept means, for example, that if displacement u_y is approximated with five terms in a particular layer k then it will be approximated with five terms in all layers of the multilayered structure.

Each displacement variable can be expanded in ∞^1 combinations. In fact, it is sufficient to change the number of terms used for each variable. Because there are three variables (the displacements u_x, u_y, and u_z), it is concluded that Eq. (9) includes ∞^3 different theories. For now the quantities are defined in a layerwise sense but it will be shown that the same concept is valid for the equivalent single layer cases as well.

Governing Equations

A multilayered structure composed of N_l layers is considered. The principle of virtual work for the case of two pressures applied at the top and bottom of each layer k is

$$\int_{\Omega^k}\int_{z_{\text{bot}_k}}^{z_{\text{top}_k}} [\delta\boldsymbol{\varepsilon}_{pG}^{kT}\boldsymbol{\sigma}_{pH}^k + \delta\boldsymbol{\varepsilon}_{nG}^{kT}\boldsymbol{\sigma}_{nH}^k]\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \delta L_e^k$$
$$= \int_{\Omega^k} \delta u_x^k(x,y,z=z_{\text{top}_k}) P_x^{kt}(x,y,z=z_{\text{top}_k})\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{\Omega^k} \delta u_y^k(x,y,z=z_{\text{top}_k}) P_y^{kt}(x,y,z=z_{\text{top}_k})\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{\Omega^k} \delta u_z^k(x,y,z=z_{\text{top}_k}) P_z^{kt}(x,y,z=z_{\text{top}_k})\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{\Omega^k} \delta u_x^k(x,y,z=z_{\text{bot}_k}) P_x^{kb}(x,y,z=z_{\text{bot}_k})\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{\Omega^k} \delta u_y^k(x,y,z=z_{\text{bot}_k}) P_y^{kb}(x,y,z=z_{\text{bot}_k})\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{\Omega^k} \delta u_z^k(x,y,z=z_{\text{bot}_k}) P_z^{kb}(x,y,z=z_{\text{bot}_k})\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{\Gamma_\sigma^k}\int_{z_{\text{bot}_k}}^{z_{\text{top}_k}} [\delta u_n^k \bar{\sigma}_{nn}^k + \delta u_s^k \bar{\sigma}_{ns}^k + \delta u_z^k \bar{\sigma}_{nz}^k]\,\mathrm{d}z\,\mathrm{d}s \tag{10}$$

where P_x^{kt}, P_y^{kt} and P_z^{kt} are the top pressures (applied at $z = z_{\text{top}_k}$), P_x^{kb}, P_y^{kb} and P_z^{kb} are the bottom pressures (applied at $z = z_{\text{bot}_k}$), $\bar{\sigma}_{nn}^k$, $\bar{\sigma}_{ns}^k$

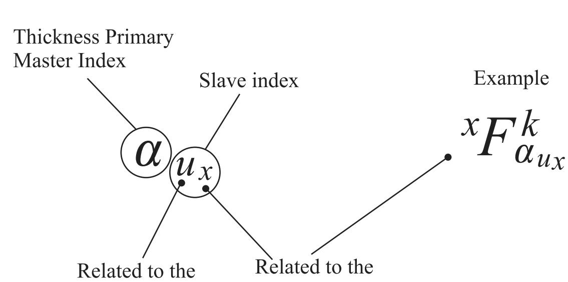


Fig. 2 GUF: master and slave indices.

and $\bar{\sigma}^k_{nz}$ are the specified normal and tangential components measured per unit area, u^k_n, u^k_s and u^k_z are the normal and tangential displacements on the edge Γ^k_σ, in which the stresses are specified. It is assumed that $\Omega^k = \Omega$.

All the stresses and strains are retained. No restricting hypotheses are formulated. The relation between the stresses and strains is Hooke's law [48] (details omitted for brevity). The geometric relations relate the strains to the derivative of the displacements. Using the formalism of the GUF it is possible to write

$$
\begin{aligned}
\varepsilon^k_{xxG} &= \frac{\partial u^k_x}{\partial x} = {}^xF_{\alpha_{u_x}}\, u^k_{x\alpha_{u_x,x}} \\
\varepsilon^k_{yyG} &= \frac{\partial u^k_y}{\partial y} = {}^yF_{\alpha_{u_y}}\, u^k_{y\alpha_{u_y,y}} \\
\gamma^k_{xyG} &= \frac{\partial u^k_x}{\partial y} + \frac{\partial u^k_y}{\partial x} = {}^xF_{\alpha_{u_x}}\, u^k_{x\alpha_{u_x,y}} + {}^yF_{\alpha_{u_y}}\, u^k_{y\alpha_{u_y,x}} \\
\gamma^k_{xzG} &= \frac{\partial u^k_z}{\partial x} + \frac{\partial u^k_x}{\partial z} = {}^zF_{\alpha_{u_z}}\, u^k_{z\alpha_{u_z,x}} + {}^xF_{\alpha_{u_x,z}}\, u^k_{x\alpha_{u_x}} \\
\gamma^k_{yzG} &= \frac{\partial u^k_z}{\partial y} + \frac{\partial u^k_y}{\partial z} = {}^zF_{\alpha_{u_z}}\, u^k_{z\alpha_{u_z,y}} + {}^yF_{\alpha_{u_y,z}}\, u^k_{y\alpha_{u_y}} \\
\varepsilon^k_{zzG} &= \frac{\partial u^k_z}{\partial z} = {}^zF_{\alpha_{u_z,z}}\, u^k_{z\alpha_{u_z}}
\end{aligned}
\tag{11}
$$

where the symbol "," indicates the derivative. For example, ${}^zF_{\alpha_{u_z,z}}$ indicates the derivative of ${}^zF_{\alpha_{u_z}}$ with respect to z. The explicit form for the function $\delta\boldsymbol{\varepsilon}^{kT}_{pG}\boldsymbol{\sigma}^k_{pH}$ under the sign of integral of Eq. (10) is (for more details about all the terms see Appendix A)

$$\delta\boldsymbol{\varepsilon}^{kT}_{pG}\boldsymbol{\sigma}^k_{pH} = \delta\varepsilon^k_{xxG}\tilde{C}^k_{12}\varepsilon^k_{yyG} + \text{other terms} \tag{12}$$

Using the geometric relations explicitly presented in Eq. (11) (for more details about all the terms see Appendix A)

$$\delta\boldsymbol{\varepsilon}^{kT}_{pG}\boldsymbol{\sigma}^k_{pH} = \tilde{C}^k_{12}\,{}^xF_{\alpha_{u_x}}\,{}^yF_{\beta_{u_y}}\,\delta u^k_{x\alpha_{u_x,x}}\,u^k_{y\beta_{u_y,y}} + \text{other terms} \tag{13}$$

The function $\delta\boldsymbol{\varepsilon}^{kT}_{nG}\boldsymbol{\sigma}^k_{nH}$ can be obtained by using a similar procedure (for more details about all the terms see Appendix A)

$$\delta\boldsymbol{\varepsilon}^{kT}_{nG}\boldsymbol{\sigma}^k_{nH} = \delta\varepsilon^k_{zzG}\tilde{C}^k_{33}\varepsilon^k_{zzG} + \text{other terms} \tag{14}$$

Using the geometric relations represented by Eq. (11) (for more details about all the terms see Appendix A):

$$\delta\boldsymbol{\varepsilon}^{kT}_{nG}\boldsymbol{\sigma}^k_{nH} = \tilde{C}^k_{33}\,{}^zF_{\alpha_{u_z,z}}\,{}^zF_{\beta_{u_z,z}}\,\delta u^k_{z\alpha_{u_z}}\,u^k_{z\beta_{u_z}} + \text{other terms} \tag{15}$$

Equations (13) and (15) [see also Eq. (10)] are integrated over the volume of layer k. The integrals along the thickness can be immediately performed. The notation used for that integrals is shown in Fig. 3. An integration by parts is performed (details omitted for brevity). Now the focus is on the external virtual work. Consider the applied pressures. They are written in a slightly different manner to use the power of the present GUF. Consider in particular the pressure term $\delta u^k_x(x, y, z_{\text{top}_k})P^{kt}_x(x, y, z_{\text{top}_k})$ [see Eq. (10)]. $P^{kt}_x(x, y, z_{\text{top}_k})$ is the force per unit of area in direction x (subscript x) and applied at the top surface ($z = z_{\text{top}_k}$) of the layer k (the superscript t indicates the top surface of layer k). The pressure $P^{kt}_x(x, y, z_{\text{top}_k})$ is thought as a function along the thickness (this is to use GUF) as follows:

$$P^{kt}_x(x, y, z_{\text{top}_k}) = {}^xF_{\alpha_{u_x}}(z = z_{\text{top}_k}) \cdot P^{kt}_{x\alpha_{u_x}}(x, y) \tag{16}$$

Notice that in Eq. (16) the same number of terms used for the expansion in the thickness direction of the displacement u^k_x has been considered. This is a natural choice, considering the fact that the pressure in x direction works with the displacement in x direction.

Now the following notation is introduced:

$${}^xF^t_{\alpha_{u_x}} \doteq {}^xF_{\alpha_{u_x}}(z = z_{\text{top}_k}) \tag{17}$$

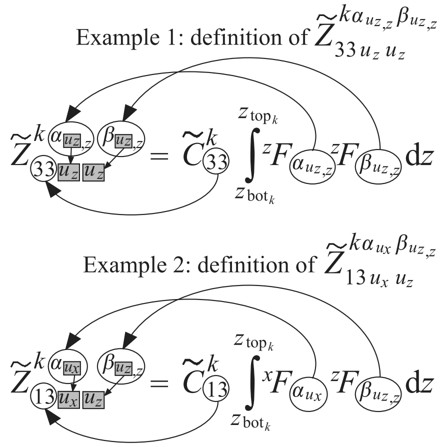


Fig. 3 GUF: examples of some definitions related to the integrals along the thickness.

The superscript t means that the functions ${}^xF_{\alpha_{u_x}}$ are all calculated at the z which corresponds to the top surface of layer k (i.e., $z = z_{\text{top}_k}$) where the pressure is assumed to be applied. Using this definition, the pressure can be written by adopting the formalism already used for the displacements

$$P^{kt}_x(x, y, z_{\text{top}_k}) = {}^xF^t_{\alpha_{u_x}} \cdot P^{kt}_{x\alpha_{u_x}}(x, y) = {}^xF^t_{\alpha_{u_x}} P^{kt}_{x\alpha_{u_x}} \tag{18}$$

Observing that

$$\delta u^k_x(x, y, z_{\text{top}_k}) = {}^xF_{\alpha_{u_x}}(z = z_{\text{top}_k}) \cdot \delta u^k_{x\alpha_{u_x}}(x, y) = {}^xF^t_{\alpha_{u_x}}\delta u^k_{x\alpha_{u_x}} \tag{19}$$

the contribution of the pressure $P^{kt}_x(x, y, z_{\text{top}_k})$ to the external virtual work can be written as (the secondary master index β needs to be used):

$$\delta u^k_x(x, y, z_{\text{top}_k})P^{kt}_x(x, y, z_{\text{top}_k}) = \delta u^k_{x\alpha_{u_x}}\,{}^xF^t_{\alpha_{u_x}}\,{}^xF^t_{\beta_{u_x}}\,P^{kt}_{x\beta_{u_x}} \tag{20}$$

or

$$\delta u^k_x(x, y, z_{\text{top}_k})P^{kt}_x(x, y, z_{\text{top}_k}) = \delta u^k_{x\alpha_{u_x}}\,{}^tD^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\,P^{kt}_{x\beta_{u_x}} \tag{21}$$

where

$${}^tD^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x} = {}^xF^t_{\alpha_{u_x}}\,{}^xF^t_{\beta_{u_x}} \tag{22}$$

The other applied pressures are similarly treated. Superscript b is used to indicate the bottom pressure; for example, ${}^xF^b_{\alpha_{u_x}} \doteq {}^xF_{\alpha_{u_x}}(z = z_{\text{bot}_k})$. The virtual external work can then be rewritten as (see Appendix B for the details about the other terms)

$$
\begin{aligned}
\delta L^k_e = &+ \int_{\Omega^k} \delta u_{x\alpha_{u_x}}\,{}^tD^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\,P^{kt}_{x\beta_{u_x}}\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{\Gamma^k_\sigma}\int_{z_{\text{bot}_k}}^{z_{\text{top}_k}} [\delta u^k_n\bar{\sigma}^k_{nn} + \delta u^k_s\bar{\sigma}^k_{ns} + \delta u^k_z\bar{\sigma}^k_{nz}]\,\mathrm{d}z\,\mathrm{d}s + \text{other terms}
\end{aligned}
\tag{23}
$$

The pressure terms (for example $P^{kt}_{x\beta_{u_x}}$) are inputs of the problem. Terms of the type ${}^tD^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x}$ are defined as pressure fundamental kernels and are 1×1 matrices.

The pressure has been treated in a particular form because it is a natural choice for the case of multilayered structures analyzed with GUF.

The term in Eq. (23), which contains the prescribed stresses, is elaborated to obtain an expression that contains the displacements u^k_x, u^k_y and u^k_z. This elaboration is required since, in general, different orders of expansions are used for different variables. After a few elaborations it can be demonstrated that the contribution of layer k to the expression of virtual external work is (see Appendix B for the details about the other terms):

$$
\begin{aligned}
\delta L_e^k &= \int_{\Omega^k} \delta u^k_{x\alpha_{u_x}}\, {}^t D^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x} P^{kt}_{x\beta_{u_x}}\, \mathrm{d}x\,\mathrm{d}y \\
&\quad + Z^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x} \int_{\Gamma^{kx}_\sigma} \delta u^k_{x\alpha_{u_x}}\, \bar{t}^k_{x\beta_{u_x}}\, \mathrm{d}s \\
&\quad + Z^{k\alpha_{u_y}\beta_{u_y}}_{u_y u_y} \int_{\Gamma^{ky}_\sigma} \delta u^k_{y\alpha_{u_y}}\, \bar{t}^k_{y\beta_{u_y}}\, \mathrm{d}s + \text{other terms}
\end{aligned} \tag{24}
$$

It is now assumed that where $\bar{t}^k_{x\alpha_{u_x}}$ is assigned the corresponding quantity $u^k_{x\alpha_{u_x}}$ is not assigned. Similarly, where $\bar{t}^k_{y\alpha_{u_y}}$ is assigned $u^k_{y\alpha_{u_y}}$ is not assigned and where $\bar{t}^k_{z\alpha_{u_z}}$ is assigned $u^k_{z\alpha_{u_z}}$ is not assigned. When the displacements are assigned, the virtual variations of the displacements are zero. For example, consider the displacement $u^k_{x\alpha_{u_x}}$, which is assigned (see above) only on the boundary portion $\Gamma^k - \Gamma^{kx}_\sigma$. Therefore, in this portion of the boundary $\delta u^k_{x\alpha_{u_x}} = 0$. Similar considerations can be made for the other displacements and, thus, the following relations can be written:

$$
\begin{aligned}
\delta u^k_{x\alpha_{u_x}} &= 0 \quad \text{on } \Gamma^k - \Gamma^{kx}_\sigma \\
\delta u^k_{y\alpha_{u_y}} &= 0 \quad \text{on } \Gamma^k - \Gamma^{ky}_\sigma \\
\delta u^k_{z\alpha_{u_z}} &= 0 \quad \text{on } \Gamma^k - \Gamma^{kz}_\sigma
\end{aligned} \tag{25}
$$

Considering these last relations and with the help of Fig. 3 the governing equations are (see Appendix C for more details about the other terms)

$$
\delta u^k_{x\alpha_{u_x}}:\; -Z^{k\alpha_{u_x}\beta_{u_x}}_{11u_x u_x} u^k_{x\beta_{u_x,x,x}} - D^{kb\alpha_{u_x}\beta_{u_x}}_{u_x u_x} P^{kb}_{x\beta_{u_x}} + \text{other terms} = 0 \tag{26}
$$

$$
\delta u^k_{y\alpha_{u_y}}:\; -Z^{k\alpha_{u_y}\beta_{u_x}}_{12u_y u_x} u^k_{x\beta_{u_x,x,y}} - D^{kb\alpha_{u_y}\beta_{u_y}}_{u_y u_y} P^{kb}_{y\beta_{u_y}} + \text{other terms} = 0 \tag{27}
$$

$$
\delta u^k_{z\alpha_{u_z}}:\; +Z^{k\alpha_{u_{z,z}}\beta_{u_x}}_{13u_z u_x} u^k_{x\beta_{u_x,x}} - D^{kb\alpha_{u_z}\beta_{u_z}}_{u_z u_z} P^{kb}_{z\beta_{u_z}} + \text{other terms} = 0 \tag{28}
$$

Boundary conditions are omitted for brevity. The governing equations here obtained are formally independent of the actual orders used to expand the displacements. This feature will lead to the invariant kernels from which all possible theories are generated.

Navier-Type Solution

To demonstrate how the invariant model is built, a Navier-type solution is considered. The invariant models are built under some hypotheses which can be easily removed when the FEM is used. The FEM procedure is not much different from a formal point of view and will be presented in future works.

In the Navier-type solution only lamination schemes with angles 0 or 90 are used. Thus, it is deduced that $\tilde{C}_{16} = \tilde{C}_{26} = \tilde{C}_{36} = \tilde{C}_{45} = 0$ [48]. Suppose also that the reference plane of the plate is a rectangle with length a in the x direction and b in the y direction. The external loads and displacements are assumed to have a sinusoidal distribution

$$
\begin{aligned}
P^{kt}_{x\beta_{u_x}} &= {}^x P^{kt}_{\beta_{u_x}} \mathcal{C}^{m\pi x}_a \mathcal{S}^{n\pi y}_b & P^{kb}_{x\beta_{u_x}} &= {}^x P^{kb}_{\beta_{u_x}} \mathcal{C}^{m\pi x}_a \mathcal{S}^{n\pi y}_b \\
P^{kt}_{y\beta_{u_y}} &= {}^y P^{kt}_{\beta_{u_y}} \mathcal{S}^{m\pi x}_a \mathcal{C}^{n\pi y}_b & P^{kb}_{y\beta_{u_y}} &= {}^y P^{kb}_{\beta_{u_y}} \mathcal{S}^{m\pi x}_a \mathcal{C}^{n\pi y}_b \\
P^{kt}_{z\beta_{u_z}} &= {}^z P^{kt}_{\beta_{u_z}} \mathcal{S}^{m\pi x}_a \mathcal{S}^{n\pi y}_b & P^{kb}_{z\beta_{u_z}} &= {}^z P^{kb}_{\beta_{u_z}} \mathcal{S}^{m\pi x}_a \mathcal{S}^{n\pi y}_b
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
u^k_{x\alpha_{u_x}} &= {}^x U^k_{\alpha_{u_x}} \mathcal{C}^{m\pi x}_a \mathcal{S}^{n\pi y}_b \qquad u^k_{y\alpha_{u_y}} = {}^y U^k_{\alpha_{u_y}} \mathcal{S}^{m\pi x}_a \mathcal{C}^{n\pi y}_b \\
u^k_{z\alpha_{u_z}} &= {}^z U^k_{\alpha_{u_z}} \mathcal{S}^{m\pi x}_a \mathcal{S}^{n\pi y}_b
\end{aligned} \tag{30}
$$

where the following definitions have been used

$$
\begin{aligned}
\mathcal{C}^{m\pi x}_a &= \cos\frac{m\pi x}{a} & \mathcal{S}^{n\pi y}_b &= \sin\frac{n\pi y}{b} \\
\mathcal{S}^{m\pi x}_a &= \sin\frac{m\pi x}{a} & \mathcal{C}^{n\pi y}_b &= \cos\frac{n\pi y}{b}
\end{aligned} \tag{31}
$$

where m and n are the wave numbers. The symbol $P^{kt}_{x\beta_{u_x}}$ which indicates the pressure (along x) at the top surface of layer k should not be confused with the amplitude ${}^x P^{kt}_{\beta_{u_x}}$ associated to the trigonometric expansion [see Eq. (29)]. Similar considerations are valid for the other pressures applied at the top and bottom surfaces of layer k. The Navier-type assumptions used for the displacements and applied pressures solve exactly the problem of simply supported plate [48]. Substituting Eqs. (29) and (30) into relations (26–28) the governing equations become

$$
\begin{aligned}
\delta u_{x\alpha_{u_x}}:&\; + K^{\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\, {}^x U_{\beta_{u_x}} + K^{\alpha_{u_x}\beta_{u_y}}_{u_x u_y}\, {}^y U_{\beta_{u_y}} + K^{\alpha_{u_x}\beta_{u_z}}_{u_x u_z}\, {}^z U_{\beta_{u_z}} = {}^x R_{\alpha_{u_x}} \\
\delta u_{y\alpha_{u_y}}:&\; + K^{\alpha_{u_y}\beta_{u_x}}_{u_y u_x}\, {}^x U_{\beta_{u_x}} + K^{\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\, {}^y U_{\beta_{u_y}} + K^{\alpha_{u_y}\beta_{u_z}}_{u_y u_z}\, {}^z U_{\beta_{u_z}} = {}^y R_{\alpha_{u_y}} \\
\delta u_{z\alpha_{u_z}}:&\; + K^{\alpha_{u_z}\beta_{u_x}}_{u_z u_x}\, {}^x U_{\beta_{u_x}} + K^{\alpha_{u_z}\beta_{u_y}}_{u_z u_y}\, {}^y U_{\beta_{u_y}} + K^{\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\, {}^z U_{\beta_{u_z}} = {}^z R_{\alpha_{u_z}}
\end{aligned} \tag{32}
$$

where the loads have been defined as follows:

$$
\begin{aligned}
{}^x R^k_{\alpha_{u_x}} &= D^{kt\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\, {}^x P^{kt}_{\beta_{u_x}} + D^{kb\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\, {}^x P^{kb}_{\beta_{u_x}} \\
{}^y R^k_{\alpha_{u_y}} &= D^{kt\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\, {}^y P^{kt}_{\beta_{u_y}} + D^{kb\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\, {}^y P^{kb}_{\beta_{u_y}} \\
{}^z R^k_{\alpha_{u_z}} &= D^{kt\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\, {}^z P^{kt}_{\beta_{u_z}} + D^{kb\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\, {}^z P^{kb}_{\beta_{u_z}}
\end{aligned} \tag{33}
$$

The nine fundamental nuclei or kernels of the GUF [see Eq. (32)] are defined as follows (here only the definition of $K^{k\alpha_{u_z}\beta_{u_z}}_{u_z u_z}$ is reported; for the remaining eight kernels see Appendix D):

$$
K^{k\alpha_{u_z}\beta_{u_z}}_{u_z u_z} = Z^{k\alpha_{u_z}\beta_{u_z}}_{55u_z u_z} \frac{m^2\pi^2}{a^2} + Z^{k\alpha_{u_z}\beta_{u_z}}_{44u_z u_z} \frac{n^2\pi^2}{b^2} + Z^{k\alpha_{u_{z,z}}\beta_{u_{z,z}}}_{33u_z u_z} \tag{34}
$$

These kernels do not change if the theory is changed or if the orders of the variables are changed. Thus, for example, a layerwise theory with cubic expansion for the in-plane displacements and parabolic expansion for the out-of-plane displacement will be generated from the same kernels used to obtain the equivalent single layer zig-zag theory that presents a parabolic expansion for the in-plane displacements and linear expansion for the out-of-plane displacement. Only six kernels are required to generate any theory. This can be seen from the following derivations. At structural level (after the assembling process) the governing equations can be written as

$$
\begin{cases}
\mathbf{K}_{u_x u_x}\, {}^x\mathbf{U} + \mathbf{K}_{u_x u_y}\, {}^y\mathbf{U} + \mathbf{K}_{u_x u_z}\, {}^z\mathbf{U} = \mathbf{D}^t_{u_x u_x}\, {}^x\mathbf{P}^t + \mathbf{D}^b_{u_x u_x}\, {}^x\mathbf{P}^b = {}^x\mathbf{P} \\
\mathbf{K}_{u_y u_x}\, {}^x\mathbf{U} + \mathbf{K}_{u_y u_y}\, {}^y\mathbf{U} + \mathbf{K}_{u_y u_z}\, {}^z\mathbf{U} = \mathbf{D}^t_{u_y u_y}\, {}^y\mathbf{P}^t + \mathbf{D}^b_{u_y u_y}\, {}^y\mathbf{P}^b = {}^y\mathbf{P} \\
\mathbf{K}_{u_z u_x}\, {}^x\mathbf{U} + \mathbf{K}_{u_z u_y}\, {}^y\mathbf{U} + \mathbf{K}_{u_z u_z}\, {}^z\mathbf{U} = \mathbf{D}^t_{u_z u_z}\, {}^z\mathbf{P}^t + \mathbf{D}^b_{u_z u_z}\, {}^z\mathbf{P}^b = {}^z\mathbf{P}
\end{cases} \tag{35}
$$

In a matrix form, the system becomes

$$
\begin{bmatrix}
\mathbf{K}_{u_x u_x} & \mathbf{K}_{u_x u_y} & \mathbf{K}_{u_x u_z} \\
 & \mathbf{K}_{u_y u_y} & \mathbf{K}_{u_y u_z} \\
\text{Symm} & & \mathbf{K}_{u_z u_z}
\end{bmatrix} \cdot
\begin{bmatrix} {}^x\mathbf{U} \\ {}^y\mathbf{U} \\ {}^z\mathbf{U} \end{bmatrix} =
\begin{bmatrix} {}^x\mathbf{P} \\ {}^y\mathbf{P} \\ {}^z\mathbf{P} \end{bmatrix} \tag{36}
$$

It is deduced [see Eq. (36)] that with the following six invariant fundamental kernels $K^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x}$, $K^{k\alpha_{u_x}\beta_{u_y}}_{u_x u_y}$, $K^{k\alpha_{u_x}\beta_{u_z}}_{u_x u_z}$, $K^{k\alpha_{u_y}\beta_{u_y}}_{u_y u_y}$, $K^{k\alpha_{u_y}\beta_{u_z}}_{u_y u_z}$ and $K^{k\alpha_{u_z}\beta_{u_z}}_{u_z u_z}$ can generate all types of theories with any order of expansion in the thickness direction. Each variable can be represented with a different order with respect the other variables. This property is also valid in the FEM representation.

The main properties of the GUF are described in Fig. 4.

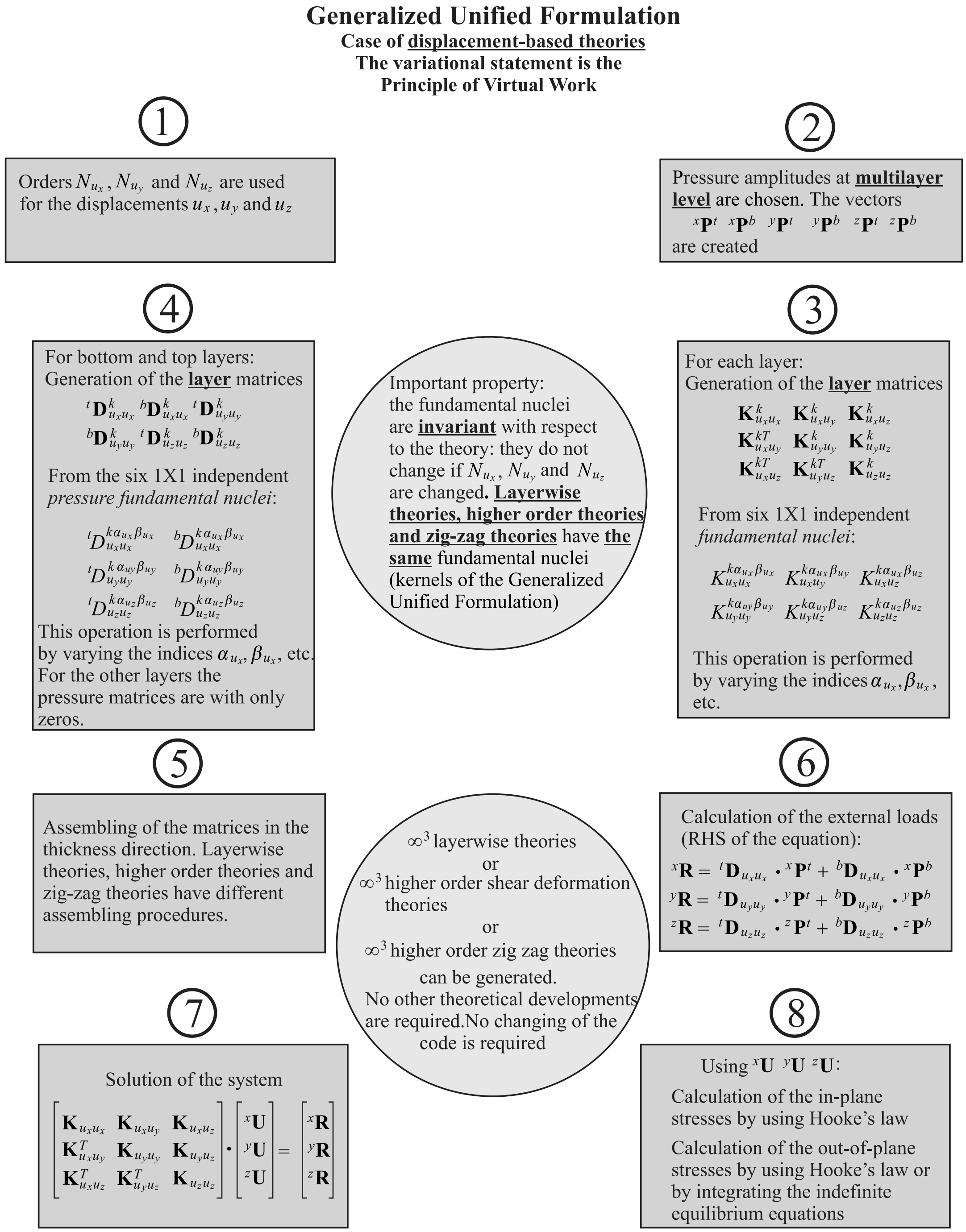


Fig. 4 GUF for displacement-based theories.

Advanced Layerwise Theories Generated by Using the GUF

Equivalent single layer theories give a sufficiently accurate description of the global laminate response. However, these theories are not adequate for determining the stress fields at ply level. Layerwise theories assume separate displacement field expansions within each layer. The accuracy is then greater but the price is in the increased computational cost. Many layerwise plate models have been proposed in the past by applying Classical Plate Theory or Higher-order Shear Deformation Theories at each layer. Generalizations of these approaches were also given, and the displacements variables were expressed in terms of Lagrange polynomials. Many papers are devoted on the subject of layerwise theories [10,11,46–51]. The conceptual differences between the displacement fields in layerwise and equivalent single layer Theories are depicted in Fig. 5. Layerwise models are computationally more expensive than the less accurate equivalent single layer models. Therefore, layerwise models can be used in regions of the structure in which an accurate description is required [53], whereas equivalent single layer models are employed in other parts of the structure. The GUF form for the displacements has been derived at layer level in Eq. (9). LWT are going to be developed and so Eq. (9) applies for this case. The functions of the thickness coordinate [see Eq. (9)] are introduced in a general form. For example, ${}^xF_t$ is a function of z and can be a polynomial, trigonometric, exponential or another function chosen a priori. To have the assembling process along the thickness direction immediate and intuitive and indicated for the case of multilayered structures a convenient expansion along the thickness is introduced.

The displacements must be continuous functions along the thickness to ensure the compatibility between two adjacent layers. Therefore, it is convenient for the axiomatic expansions along the thickness to have the following properties. The first property is that

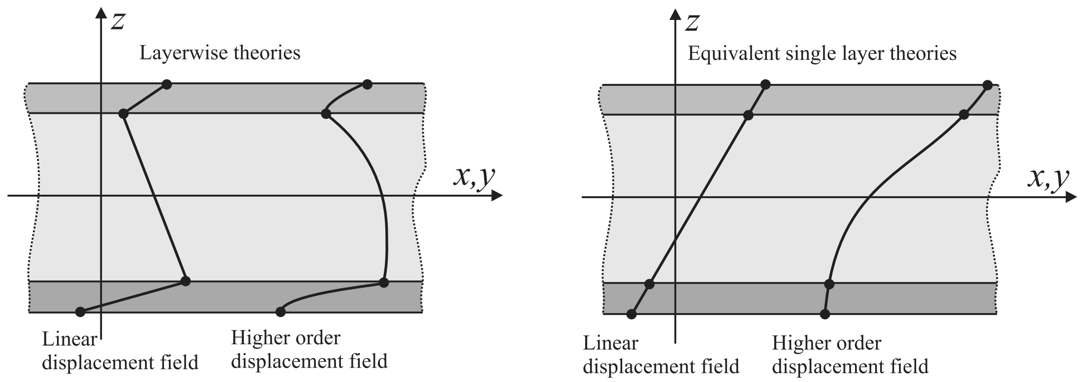


Fig. 5 Layerwise theories vs equivalent single layer theories in a three layered structure.

for $z = z_{\mathrm{bot}_k}$ (the bottom surface of layer k) all the functions along the thickness are zero except the one which multiplies the term corresponding to the bottom (subscript b). For example, in the case of the displacement u_x^k, the functions calculated at the bottom of layer k should give the following values:

$$ {}^xF_t(z = z_{\mathrm{bot}_k}) = 0 \qquad {}^xF_l(z = z_{\mathrm{bot}_k}) = 0 \qquad {}^xF_b(z = z_{\mathrm{bot}_k}) = 1 \tag{37} $$

If the previous conditions are satisfied then $u_{x_b}^k$ is not just a term in the thickness expansion of the variable u_x^k but assumes the meaning of the value that the displacement u_x^k takes when the bottom surface of layer k is considered (i.e., $z = z_{\mathrm{bot}_k}$). This now explains why the subscript "b" is introduced in the notation. Similar concept is applied for the top surface of the layer. A good set of function satisfies the following relations:

$$ {}^xF_t(z = z_{\mathrm{top}_k}) = 1 \qquad {}^xF_l(z = z_{\mathrm{top}_k}) = 0 \qquad {}^xF_b(z = z_{\mathrm{top}_k}) = 0 \tag{38} $$

and so $u_{x_t}^k$ is not just a term in the thickness expansion of the variable u_x^k but assumes the meaning of the value that the displacement u_x^k takes when the top surface of layer k is considered (i.e., $z = z_{\mathrm{top}_k}$). This now explains why the subscript t is introduced. The second property is that numerical stability (no ill conditioning when the orders are increased) is important. Therefore, orthogonal polynomials should be preferred.

A good set of functions which satisfy the above mentioned properties should be selected. It is possible to demonstrate that all the previous properties are satisfied if particular combination of Legendre polynomials is used. Legendre polynomials are defined in the interval [−1, 1]. Thus, a transformation is necessary:

$$ \zeta_k = \frac{2}{z_{\mathrm{top}_k} - z_{\mathrm{bot}_k}} z - \frac{z_{\mathrm{top}_k} + z_{\mathrm{bot}_k}}{z_{\mathrm{top}_k} - z_{\mathrm{bot}_k}} \qquad -1 \le \zeta_k \le +1 \tag{39} $$

where ζ_k is a nondimensional coordinate. The Legendre polynomial of order zero is $P_0(\zeta_k) = 1$. The Legendre polynomial of order one is $P_1(\zeta_k) = \zeta_k$. The higher-order polynomials can be obtained by using Bonnet's recursion [55] formula. Bonnet's formula is a convenient method to calculate the Legendre polynomials in a practical code based on the GUF.

The same type of functions for all displacements are used. This is not necessary with the GUF but it is more practical. The following combination of Legendre functions is used:

$$ \begin{aligned} &{}^xF_t = {}^yF_t = {}^zF_t = \frac{P_0 + P_1}{2}, \qquad {}^xF_b = {}^yF_b = {}^zF_b = \frac{P_0 - P_1}{2} \\ &{}^xF_l = P_l - P_{l-2}, \qquad l = 2, 3, \ldots, N_{u_x} \\ &{}^yF_m = P_m - P_{m-2}, \qquad m = 2, 3, \ldots, N_{u_y} \\ &{}^zF_n = P_n - P_{n-2}, \qquad n = 2, 3, \ldots, N_{u_z} \end{aligned} \tag{40} $$

in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of j-order. The chosen functions have the following properties:

$$ \zeta_k = \begin{cases} +1, & {}^xF_t, {}^yF_t, {}^zF_t = 1, {}^xF_b, {}^yF_b, {}^zF_b = 0, {}^xF_l, {}^yF_m, {}^zF_n = 0 \\ -1, & {}^xF_t, {}^yF_t, {}^zF_t = 0, {}^xF_b, {}^yF_b, {}^zF_b = 1, {}^xF_l, {}^yF_m, {}^zF_n = 0 \end{cases} \tag{41} $$

Thus, the properties earlier mentioned are all satisfied and this set of functions is a good choice to build the advanced layerwise theories. It is then possible to create any class of theories by changing the orders of displacements. Suppose, for example, that a theory has the following data: $N_{u_x} = 3$, $N_{u_y} = 2$, $N_{u_z} = 4$. The corresponding theory is indicated as LD_{324}. The first letter L means layerwise theory, the second letter D means that a displacement-based theory is formulated (i.e., the variational statement is the principle of virtual displacements) The subscripts are the orders of the Legendre polynomials used for the displacements. In general, the acronym is then built as follows: $LD_{N_{u_x} N_{u_y} N_{u_z}}$.

Expansion of the Six 1 × 1 Invariant Kernels: Matrices at Layer Level

In this paper the expansion used in the different variables does not change and each layer is treated in the same way. Thus, for example, $N_{u_x}^k = N_{u_x}^{k+1} = N_{u_x}$. The expansion of the kernels is the most important part of the generation of one of the possible ∞^3 layerwise theories. This operation is done at layer level. To explain how this operation is performed, consider the case of theory LD_{326}, in which the number of degrees of freedom, *at layer level*, is the following:

$$ \begin{aligned} [\mathrm{DOF}]_{u_x}^k &= N_{u_x} + 1 = 3 + 1 = 4 \\ [\mathrm{DOF}]_{u_y}^k &= N_{u_y} + 1 = 2 + 1 = 3 \\ [\mathrm{DOF}]_{u_z}^k &= N_{u_z} + 1 = 6 + 1 = 7 \end{aligned} \tag{42} $$

From the number of degrees of freedom it is possible to calculate the size of the layer matrices. For example, when matrix $K_{u_x u_z}^{k\alpha_{u_x}\beta_{u_z}}$ is expanded then the final size at layer level will be $[\mathrm{DOF}]_{u_x}^k \times [\mathrm{DOF}]_{u_z}^k$. In the example relative to theory LD_{326}, matrix $K_{u_x u_z}^{k\alpha_{u_x}\beta_{u_z}}$ at layer level (indicated as $\mathbf{K}_{u_x u_z}^k$) is a 4×7 matrix and obtained as explained in Fig. 6.

Assembling in the Thickness Direction: From Layer to Multilayer Level

The assembling must consider the compatibility of the displacements between two adjacent layers. Figure 7 shows how the assembling of a typical matrix is performed. The pressure matrices are obtained from the pressure kernels using the same method shown in Figs. 6 and 7. The use of combinations of Legendre polynomials ensures the continuity of the functions, which are used to expand the displacements. Figure 8 shows this concept for the case of theory

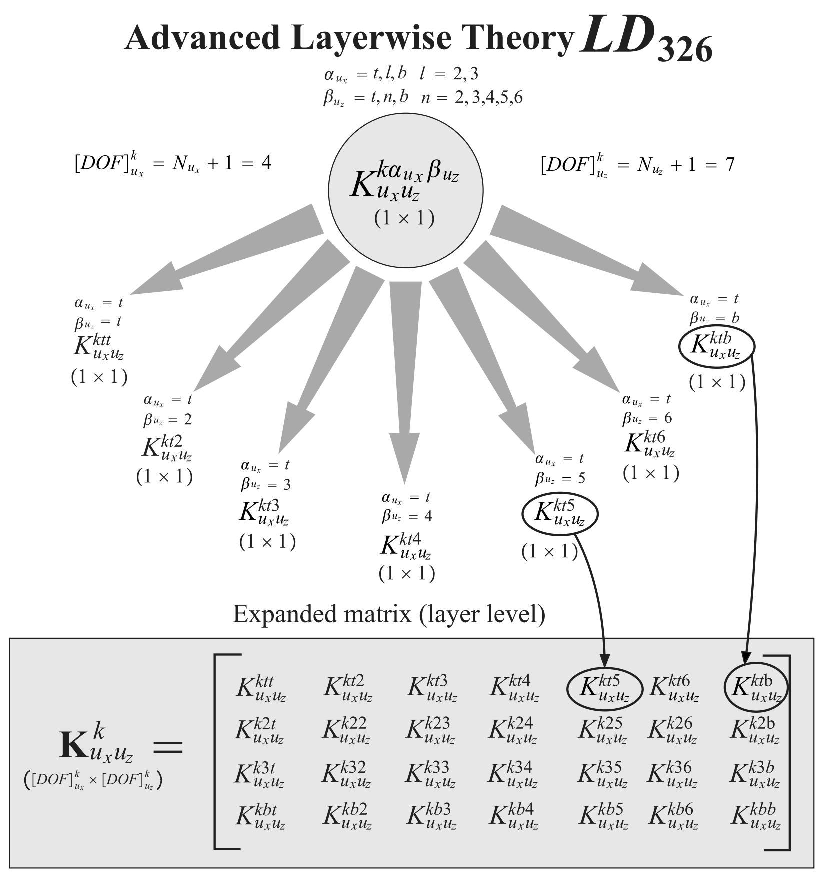


Fig. 6 GUF: example of expansion from a kernel to a layer matrix. Case of theory LD_{326}, from $K^{k\alpha_{u_x}\beta_{u_z}}_{u_x u_z}$ to $\mathbf{K}^k_{u_x u_z}$.

LD_{324}. The pressure amplitudes at multilayer level are input of the problem. The layer pressure matrices (for example ${}^t\mathbf{D}^k_{u_x u_x}$) are obtained using a method similar to the one adopted for the stiffness matrices. They are generated from the pressure fundamental kernels and assembled. In the practice, all the pressure matrices are set to be with only zeros except the top and bottom layers in which the pressure is actually applied. Once the matrices are all assembled, Eq. (36) is obtained and the displacement amplitudes can be found. In the case of FEM applications the unknowns will be the nodal displacements and the concepts are the same.

Advanced Higher-Order Shear Deformation Theories Generated by Using the GUF

The derivation is started by considering a particular theory in which the in-plane displacements are expanded along the thickness by using a cubic polynomial and the out-of-plane displacement u_z is parabolic. In this case it is possible to write the displacements as follows:

$$\text{Theory I}: \begin{cases} u_x = u_{x_0} + z\phi_{1_{u_x}} + z^2\phi_{2_{u_x}} + z^3\phi_{3_{u_x}} \\ u_y = u_{y_0} + z\phi_{1_{u_y}} + z^2\phi_{2_{u_y}} + z^3\phi_{3_{u_y}} \\ u_z = u_{z_0} + z\phi_{1_{u_z}} + z^2\phi_{2_{u_z}} \end{cases} \tag{43}$$

For each displacement component the concepts of the GUF can be applied. For example, the displacement u_x is written as

$$\begin{aligned} u_x &= u_{x_0} + z\phi_{1_{u_x}} + z^2\phi_{2_{u_x}} + z^3\phi_{3_{u_x}} \\ &= {}^xF_t u_{x_t} + {}^xF_2 u_{x_2} + {}^xF_3 u_{x_3} + {}^xF_b u_{x_b} \end{aligned} \tag{44}$$

where

$$\begin{aligned} &{}^xF_t = 1; \qquad {}^xF_2 = z; \qquad {}^xF_3 = z^2; \qquad {}^xF_b = z^3 \\ &u_{x_t} = u_{x_0}; \qquad u_{x_2} = \phi_{1_{u_x}}; \qquad u_{x_3} = \phi_{2_{u_x}}; \qquad u_{x_b} = \phi_{3_{u_x}} \end{aligned} \tag{45}$$

The GUF for the displacement u_x is

$$u_x = {}^xF_{\alpha_{u_x}} u_{x\alpha_{u_x}} \qquad \alpha_{u_x} = t, l, b; \qquad l = 2, \ldots, N_{u_x} \tag{46}$$

where, in the example (see Eqs. (43), (44), and (46), $N_{u_x} = 3$. Notice that the superscript x in ${}^xF_{\alpha_{u_x}}$ is used to clearly enhance that the displacement u_x (displacement in the x direction) is being considered.

For the displacement u_y the formal procedure produces similar results (notice that now the slave index is u_y):

$$u_y = {}^yF_{\alpha_{u_y}} u_{y\alpha_{u_y}} \qquad \alpha_{u_y} = t, m, b; \qquad m = 2, \ldots, N_{u_y} \tag{47}$$

The displacement u_z is only parabolic [three terms are used in the expansion; see Eq. (43)], but the representation is formally the same:

$$u_z = {}^zF_{\alpha_{u_z}} u_{z\alpha_{u_z}} \qquad \alpha_{u_z} = t, n, b; \qquad n = 2, \ldots, N_{u_z} \tag{48}$$

The superscript k is not present, whereas in the layerwise case it was. In fact, in equivalent single layer models the displacement fields have a description at plate level and not at layerwise level. The GUF for this case (Eqs. (46–48)) is formally equivalent to the writing of Eq. (9). This similarity suggests that it is possible to use the layerwise GUF for the equivalent single layer case as well. That is, Eq. (9) can be used for the equivalent single layer case. The fact that the displacement field does not have a layerwise description (see, for example, Theory I explicitly written in Eq. (43)) is taken into account when the assembling in the thickness direction of the layer matrices is

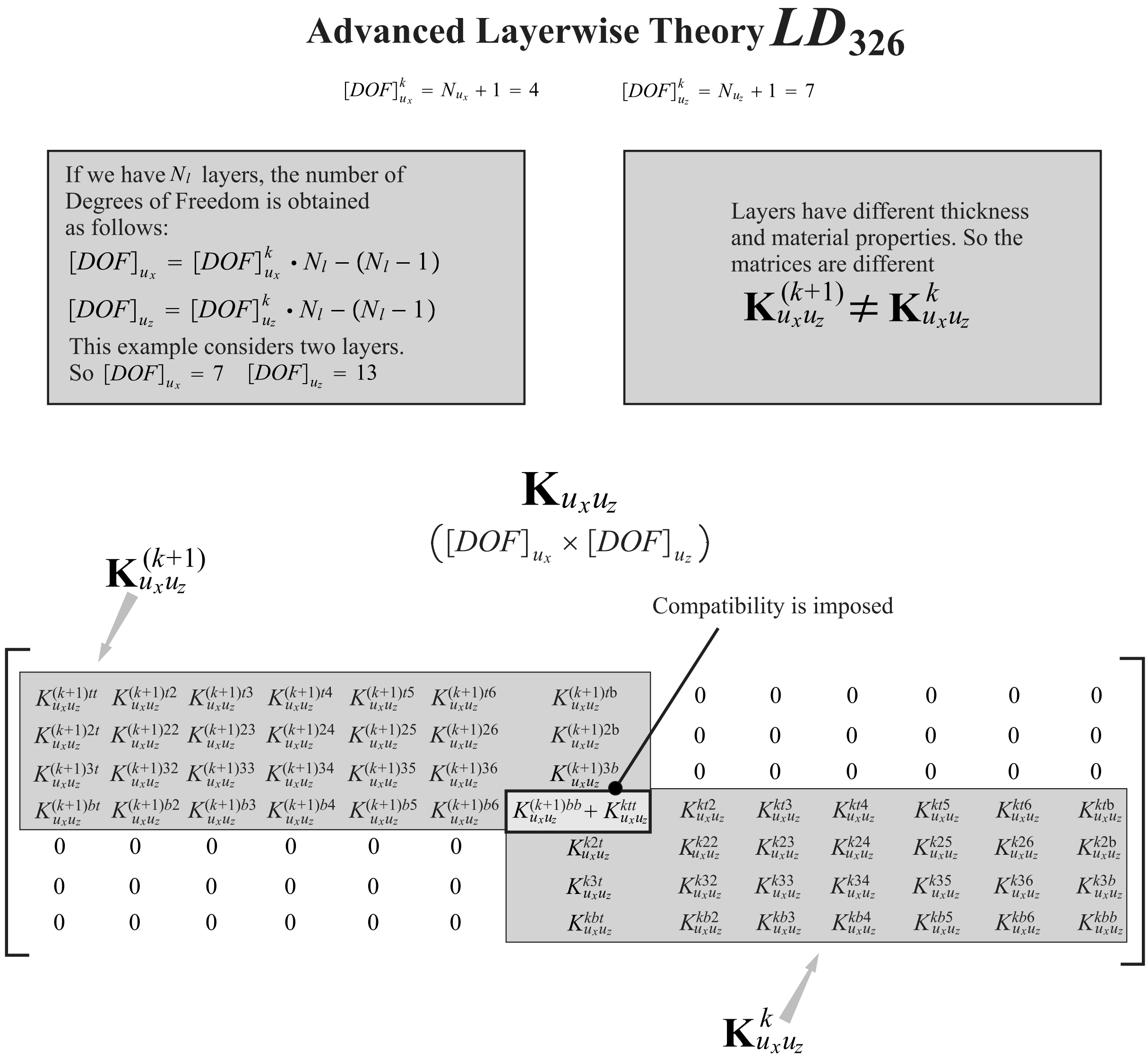


Fig. 7 GUF: example of assembling from layer matrices to multilayer matrix. Case of theory LD_{326}, from $\mathbf{K}^{k}_{u_x u_z}$ and $\mathbf{K}^{(k+1)}_{u_x u_z}$ to $\mathbf{K}_{u_x u_z}$.

considered. For the AHSDTs it is then possible to use Eq. (9). However, now the functions of the thickness coordinates are going to be different. The functions of the thickness coordinate are assumed to be of the type $1, z, z^2, z^3 \ldots$. This choice is made for consistency with the usual approach used in the literature for equivalent single layer models. The actual functions are reported below:

$$
\begin{aligned}
&{}^xF_t = 1 \qquad {}^yF_t = 1 \qquad {}^zF_t = 1\\
&{}^xF_2 = z \qquad {}^yF_2 = z \qquad {}^zF_2 = z\\
&{}^xF_3 = z^2 \qquad {}^yF_3 = z^2 \qquad {}^zF_3 = z^2\\
&\ldots\\
&{}^xF_l = z^{l-1} \qquad {}^yF_m = z^{m-1} \qquad {}^zF_n = z^{n-1}\\
&\ldots\\
&{}^xF_b = z^{N_{u_x}} \qquad {}^yF_b = z^{N_{u_y}} \qquad {}^zF_b = z^{N_{u_z}}
\end{aligned}
\tag{49}
$$

As for the layerwise case it is possible to create a class of theories by changing the order used for the displacements. Suppose, for example, that a theory has the following data: $N_{u_x} = 3$, $N_{u_y} = 2$, $N_{u_z} = 4$. The corresponding advanced higher-order shear deformation theory is indicated as ED_{324}. The first letter E means equivalent single layer theory, the second letter D means that a displacement-based formulation is used. The subscripts are the orders of the polynomials used for the displacements. In general, the acronym is then built as follows: $ED_{N_{u_x} N_{u_y} N_{u_z}}$.

Expansion of the Six 1 × 1 Invariant Kernels: Matrices at Layer Level

Besides the different form of the functions used to expand the displacements, AHSDTs do not present formal differences with respect to the layerwise cases. For example, the generation of layer matrix $\mathbf{K}^{k}_{u_x u_z}$ from the kernel $K^{k\alpha_{u_x}\beta_{u_z}}_{u_x u_z}$ in the cases of theories LD_{326} and ED_{326} (which have the same number of degrees of freedom at layer level) is formally the same. Thus, Fig. 6 applies for the case of theory ED_{326} as well. Of course, the actual integrals along the thickness are different because different functions have been used (compare Eqs. (40) and (49) and see the definitions of Fig. 3).

Assembling in the Thickness Direction: From Layer to Multilayer Level

The displacement fields are treated as equivalent single layer quantities, and this makes the assembling procedure different with respect to the layerwise case. The continuity of the displacements in the thickness direction must be imposed. So, it is not difficult to show that the assembling is performed as shown in Fig. 9). The pressure

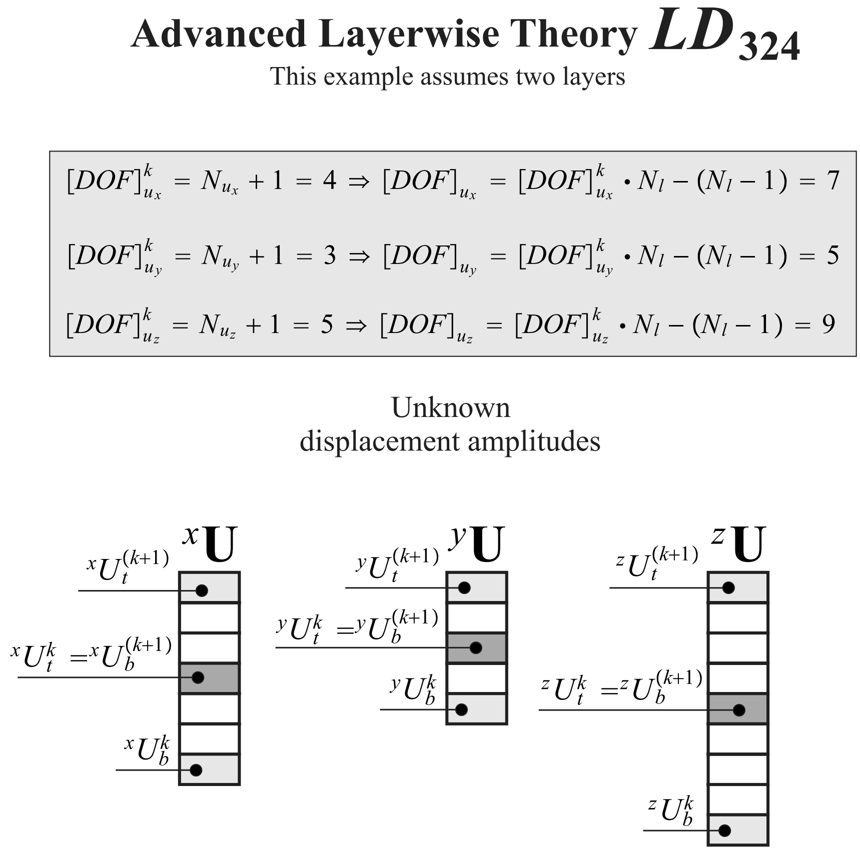


Fig. 8 Case of theory LD_{324}. Multilayer unknown displacement and amplitudes for the case in which the number of layers is two.

matrices are obtained using similar procedure and the details are omitted for brevity. The final equations at multilayer level are again given by relation (36). This equivalence is another advantage of the GUF.

AHSDTs with Zig–Zag Effects Generated by Using the GUF

The higher-order shear deformation theories may be not sufficiently accurate in some challenging cases. One option would be to abandon the equivalent single layer description and use layerwise theories. However, this may be computationally too expensive. Is there anything more accurate than HSDT with a similar number of degrees of freedom? The answer is yes, zig–zag theories [43]. The concept behind zig–zag theories and zig–zag form of the displacements is the following. The equilibrium between two adjacent layers implies that the out-of-plane stresses are equal at the interface. These stresses can be thought as a combination of strains multiplied by some coefficients that depend on the material of each layer (Hooke's law). In general, two layers have different mechanical properties and, therefore, different strains are required to obtain equilibrium. The strains are related to the derivatives of the displacements (geometric relations). Thus, different strains imply different slopes of the displacements. This fact leads to the zig–zag form of the displacements (see Fig. 10).

A very large amount of literature has been devoted to the formulation of axiomatic zig–zag theories that take into account these requirements. Three different categories [43] of zig–zag theories can be created. The first category is Lekhnitskii Multilayered Theory (LMT). The second is Ambartsumian Multilayered Theory (AMT) and the third is Reissner Multilayered Theory (RMT). LMT was introduced for the particular case of cantilevered multilayered beam [32] and almost ignored in subsequent works with a few exceptions [37,38]. A summary of the main facts of LMT is presented in [43]. Ambartsumian work [33–36] was an extension of Reissner–Mindlin theory [28,29]. RMT is based on Reissner's mixed variational theorem [56,57].

The present paper takes into account the zig–zag effects by adopting Murakami's zig–zag function [31] (see Fig. 10). MZZF has the advantage of being simple and reproducing the discontinuity of the first derivative of the displacements in the thickness direction. It will be demonstrated that the usage of MZZF is more effective than increasing the orders of the expansions of the variables along the thickness.

The derivation is started from the advanced higher-order shear deformation theory expressed by Eq. (43). Theory I [see Eq. (43)] is then improved as follows (Theory II is the resulting theory):

Theory II:

$$\begin{cases} u_x = u_{x_0} + z\phi_{1_{u_x}} + z^2\phi_{2_{u_x}} + z^3\phi_{3_{u_x}} + \overbrace{(-1)^k \zeta_k u_{x_Z}}^{\text{Extra term for Zig-Zag effect}} \\ u_y = u_{y_0} + z\phi_{1_{u_y}} + z^2\phi_{2_{u_y}} + z^3\phi_{3_{u_y}} + \overbrace{(-1)^k \zeta_k u_{y_Z}}^{\text{Extra term for Zig-Zag effect}} \\ u_z = u_{z_0} + z\phi_{1_{u_z}} + z^2\phi_{2_{u_z}} + \overbrace{(-1)^k \zeta_k u_{z_Z}}^{\text{Extra term for Zig-Zag effect}} \end{cases} \tag{50}$$

The quantity ζ_k is defined in Eq. (39). In Eq. (50), valid for a theory with Zig–Zag form of the displacements included, the following can be observed.

In Murakami's zig–zag functions the term $(-1)^k$ is present. k is the integer representing the ID of a generic layer, $k = 1$ is for the bottom layer and $k = N_l$ is for the top layer (N_l is the number of layers). The term $(-1)^k$ enforces the discontinuity of the first derivative (thickness direction) of the displacement. For example, in layer k the derivative with respect to z of the zig–zag term relative to the component u_x is

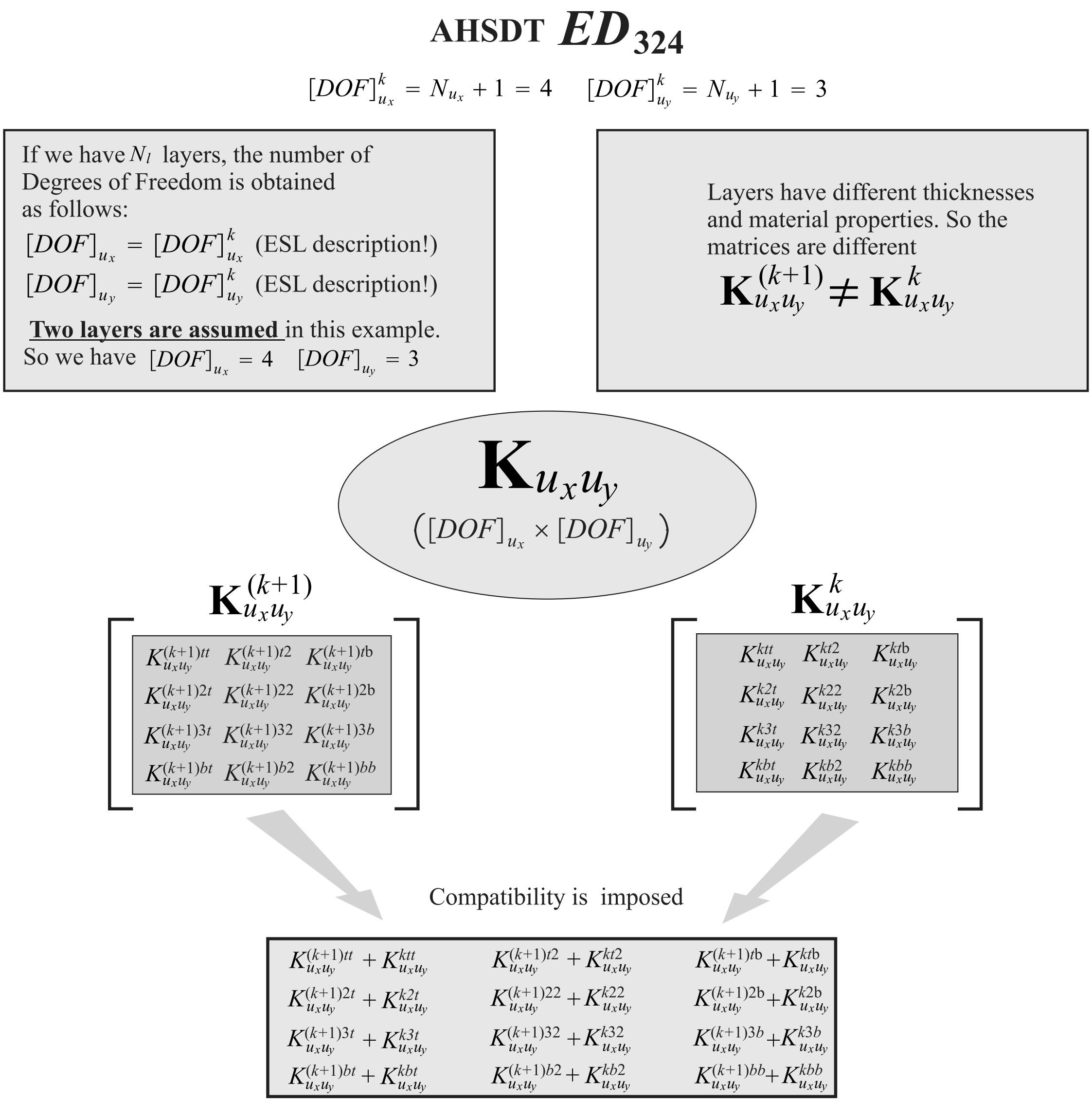


Fig. 9 GUF: example of assembling from layer matrices to multilayer matrix. Case of theory ED_{324}, from $\mathbf{K}^k_{u_xu_y}$ and $\mathbf{K}^{(k+1)}_{u_xu_y}$ to $\mathbf{K}_{u_xu_y}$.

$$\frac{\mathrm{d}[(-1)^k\zeta_k u_{x_Z}]}{\mathrm{d}z} = (-1)^k u_{x_Z}\frac{\mathrm{d}\zeta_k}{\mathrm{d}z} = (-1)^k u_{x_Z}\frac{2}{z_{\mathrm{top}_k} - z_{\mathrm{bot}_k}} \tag{51}$$

As can be seen the term $(-1)^k$ strongly affects the sign of the derivative. The displacements still have an equivalent single layer description. In fact, the terms u_{x_Z}, u_{y_Z} and u_{z_Z} are independent of the actual layer and defined for the whole plate. The zig–zag form of the displacements is taken into account a priori by adding only 3 degrees of freedom (u_{x_Z}, u_{y_Z} and u_{z_Z}, respectively). This is a general property and does not depend on the orders used for the expansion of the different variables. That is, a generic theory can take into account the zig–zag form of the displacements by adding only three extra degrees of freedoms, as was done in Eqs. (43) and (50) for the case of Theory I. For each displacement component the concepts of the GUF is still valid. However, in this case Eqs. (46–48) have to be slightly modified because they have to contain an extra term which comes from the zig–zag function. This is explained if for example displacement u_x is considered

$$\begin{aligned} u_x &= u_{x_0} + z\phi_{1_{u_x}} + z^2\phi_{2_{u_x}} + z^3\phi_{3_{u_x}} + (-1)^k\zeta_k u_{x_Z} \\ &= {}^xF_t u_{x_t} + {}^xF_2 u_{x_2} + {}^xF_3 u_{x_3} + {}^xF_4 u_{x_4} + {}^xF_b u_{x_b} \end{aligned} \tag{52}$$

where

$$\begin{aligned} &{}^xF_t = 1; \quad {}^xF_2 = z; \quad {}^xF_3 = z^2; \\ &{}^xF_4 = z^3; \quad {}^xF_b = (-1)^k\zeta_k \\ &u_{x_t} = u_{x_0}; \quad u_{x_2} = \phi_{1_{u_x}}; \quad u_{x_3} = \phi_{2_{u_x}}; \\ &u_{x_4} = \phi_{3_{u_x}}; \quad u_{x_b} = u_{x_Z} \end{aligned} \tag{53}$$

The GUF for the displacement u_x is

$$u_x = {}^xF_{\alpha_{u_x}} u_{x\alpha_{u_x}} \qquad \alpha_{u_x} = t, l, b; \qquad l = 2, \ldots, N_{u_x} + 1 \tag{54}$$

where, in the example, $N_{u_x} = 3$. For the displacement u_y the formal procedure produces similar results:

$$u_y = {}^yF_{\alpha_{u_y}} u_{y\alpha_{u_y}} \qquad \alpha_{u_y} = t, m, b; \qquad m = 2, \ldots, N_{u_y} + 1 \tag{55}$$

The displacement u_z is only parabolic, but the representation is formally the same

$$u_z = {}^zF_{\alpha_{u_z}} u_{z\alpha_{u_z}} \qquad \alpha_{u_z} = t, n, b; \qquad n = 2, \ldots, N_{u_z} + 1 \tag{56}$$

For the case of zig–zag theories, the GUF [Eqs. (54–56)] is formally equivalent to a layerwise case in which the same number of terms is considered. Thus see Eq. (9)

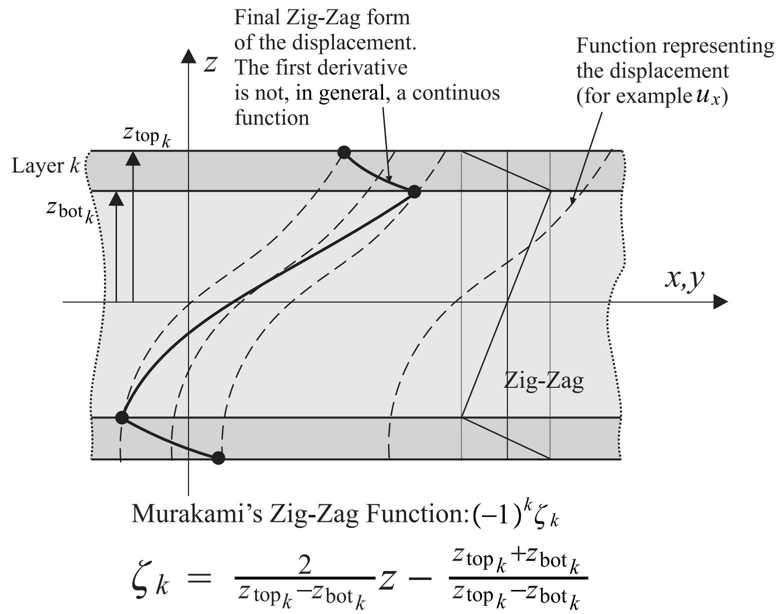


Fig. 10 Zig–zag form of the displacements and Murakami's zig–zag function.

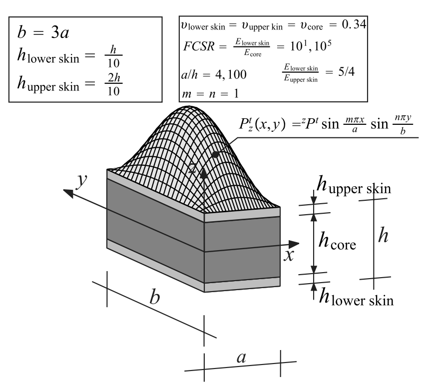


Fig. 11 Geometry of the plate sandwich structure.

$$
\begin{aligned}
u_x^k &= {}^xF_{\alpha_{u_x}} u^k_{x\alpha_{u_x}} \qquad \alpha_{u_x} = t, l, b; \qquad l = 2, \dots, N_{u_x} + 1 \\
u_y^k &= {}^yF_{\alpha_{u_y}} u^k_{y\alpha_{u_y}} \qquad \alpha_{u_y} = t, m, b; \qquad m = 2, \dots, N_{u_y} + 1 \\
u_z^k &= {}^zF_{\alpha_{u_z}} u^k_{z\alpha_{u_z}} \qquad \alpha_{u_z} = t, n, b; \qquad n = 2, \dots, N_{u_z} + 1
\end{aligned}
\tag{57}
$$

where

$$
\begin{aligned}
&{}^xF_t = 1 \qquad {}^yF_t = 1 \qquad {}^zF_t = 1 \\
&{}^xF_2 = z \qquad {}^yF_2 = z \qquad {}^zF_2 = z \\
&{}^xF_3 = z^2 \qquad {}^yF_3 = z^2 \qquad {}^zF_3 = z^2 \\
&\dots \\
&{}^xF_l = z^{l-1} \qquad {}^yF_m = z^{m-1} \qquad {}^zF_n = z^{n-1} \\
&\dots \\
&{}^xF_b = (-1)^k \zeta_k \qquad {}^yF_b = (-1)^k \zeta_k \qquad {}^zF_b = (-1)^k \zeta_k
\end{aligned}
\tag{58}
$$

The AZZTs are obtained by adding the zig–zag form of the displacements to the AHSDTs presented above. This is also evident if Eqs. (49) and (58) are compared. It is possible to create a class of theories by changing the order used for the displacements. Suppose, for example, that a theory has the following data: $N_{u_x} = 3$, $N_{u_y} = 2$, $N_{u_z} = 4$. The corresponding AZZT theory is indicated as EDZ_{324}. The first letter E means equivalent single layer theory, the second letter D means displacement-based theory, the third letter Z means that the zig–zag form of the displacements is enforced a priori. The subscripts are the orders of the polynomials used for the displacements. In general the acronym is then built as follows: $EDZ_{N_{u_x} N_{u_y} N_{u_z}}$. The formal generation of the stiffness matrices from the kernels of the GUF is done as in the case of AHSDTs. The difference is only in the sizes of the matrices and of course in the actual values of the integrals along the thickness. For example, consider theories ED_{324} and EDZ_{324}. They both have the same orders. However, in the zig–zag theory there is an extra degree of freedom for each displacement variable. This means that the layer matrix $\mathbf{K}^k_{u_x u_y}$ has size 4×3 in the case of theory ED_{324} whereas the size is 5×4 in the case of theory EDZ_{324}. The assembling at multilayered level follows the same procedure outlined in the case of AHSDTs. The final system that has to be solved is again Eq. (36).

Results

The multilayered structure is a sandwich plate (see Fig. 11) made of two skins and a core [$h_{\text{lower skin}} = h/10$; $h_{\text{upper skin}} = 2h/10$; $h_{\text{core}} = (7/10)h$]. It is also $\frac{E_{\text{lower skin}}}{E_{\text{upper skin}}} = 5/4$. The plate is simply supported and the load is a sinusoidal pressure applied at the top surface of the plate ($m = n = 1$). Different cases are proposed

Table 1 Comparison of various layerwise theories to evaluate the transverse displacement amplitude (center plate deflection) $\hat{u}_z = u_z \frac{100 E_{\text{core}}}{{}^zP^t h (\frac{a}{h})^4}$ in $z = z^{\text{upper skin}}_{\text{bottom}} = \frac{3}{10} h, x = a/2, y = b/2$

a/h	4		100		
			FCSR = 10^1		
Elasticity	3.01123	Error %	1.51021	Error %	DOF
LD_{111}	2.98058	(−1.02)	1.47242	(−2.50)	12
LD_{115}	3.00603	(−0.17)	1.51020	(0.00)	24
LD_{225}	3.00983	(−0.05)	1.51021	(0.00)	30
LD_{555}	3.01123	(0.00)	1.51021	(0.00)	48
			FCSR = 10^5		
Elasticity	$1.31593 \cdot 10^{-02}$	Err.%	$2.08948 \cdot 10^{-03}$	Err.%	
LD_{111}	$9.79008 \cdot 10^{-03}$	(−25.6)	$1.96509 \cdot 10^{-03}$	(−5.95)	12
LD_{115}	$1.31222 \cdot 10^{-02}$	(−0.28)	$2.08948 \cdot 10^{-03}$	(0.00)	24
LD_{225}	$1.31473 \cdot 10^{-02}$	(−0.09)	$2.08948 \cdot 10^{-03}$	(0.00)	30
LD_{555}	$1.31593 \cdot 10^{-02}$	(0.00)	$2.08949 \cdot 10^{-03}$	(0.00)	48

here: 1) face-to-core stiffness ratio = FCSR = $\frac{E_{\text{lower skin}}}{E_{\text{core}}} = 10^1$; $a/h = 4, 100$, and 2) face-to-core stiffness ratio = FCSR= $\frac{E_{\text{lower skin}}}{E_{\text{core}}} = 10^5$; $a/h = 4, 100$.

As far as Poisson's ratio is concerned, the following values are used: $\upsilon_{\text{lower skin}} = \upsilon_{\text{upper skin}} = \upsilon_{\text{core}} = \upsilon = 0.34$. In all cases $b = 3a$. In this test case there is no symmetry with respect the plane $z = 0$. The following nondimensional quantities are introduced:

$$\hat{u}_x = u_x \frac{E_{\text{core}}}{zP^t h(\frac{a}{h})^3}; \quad \hat{u}_y = u_y \frac{E_{\text{core}}}{zP^t h(\frac{a}{h})^3}; \quad \hat{u}_z = u_z \frac{100E_{\text{core}}}{zP^t h(\frac{a}{h})^4};$$
$$\hat{\sigma}_{zx} = \frac{\sigma_{zx}}{zP^t(\frac{a}{h})}; \quad \hat{\sigma}_{zy} = \frac{\sigma_{zy}}{zP^t(\frac{a}{h})}; \quad \hat{\sigma}_{zz} = \frac{\sigma_{zz}}{zP^t};$$
$$\hat{\sigma}_{xx} = \frac{\sigma_{xx}}{zP^t(\frac{a}{h})^2}; \quad \hat{\sigma}_{yy} = \frac{\sigma_{yy}}{zP^t(\frac{a}{h})^2}; \quad \hat{\sigma}_{xy} = \frac{\sigma_{xy}}{zP^t(\frac{a}{h})^2}; \tag{59}$$

Table 2 Comparison of various AHSDT to evaluate the transverse displacement amplitude (center plate deflection) $\hat{u}_z = u_z \frac{100E_{\text{core}}}{zP^t h(\frac{a}{h})^4}$ in $z = z_{\text{bottom}}^{\text{upper skin}} = \frac{3}{10}h, x = a/2, y = b/2$

a/h	4		100		
		FCSR = 10^1			
Elasticity	3.01123	Err.%	1.51021	Err.%	DOF
ED_{111}	1.58218	(−47.5)	1.10845	(−26.6)	6
ED_{115}	2.00048	(−33.6)	1.50862	(−0.11)	10
ED_{225}	2.03295	(−32.5)	1.50864	(−0.10)	12
ED_{335}	2.73658	(−9.12)	1.50979	(−0.03)	14
ED_{444}	2.79960	(−7.03)	1.50989	(−0.02)	15
ED_{445}	2.80707	(−6.78)	1.50989	(−0.02)	16
ED_{555}	2.84978	(−5.36)	1.50996	(−0.02)	18
ED_{777}	2.86875	(−4.73)	1.50999	(−0.01)	24
		FCSR = 10^5			
Elasticity	$1.31593 \cdot 10^{-02}$	Err.%	$2.08948 \cdot 10^{-03}$	Err.%	
ED_{111}	$1.79831 \cdot 10^{-04}$	(−98.6)	$1.19941 \cdot 10^{-04}$	(−94.3)	6
ED_{115}	$2.49683 \cdot 10^{-04}$	(−98.1)	$1.63241 \cdot 10^{-04}$	(−92.2)	10
ED_{225}	$4.09818 \cdot 10^{-04}$	(−96.9)	$1.63247 \cdot 10^{-04}$	(−92.2)	12
ED_{335}	$7.83554 \cdot 10^{-04}$	(−94.0)	$1.64006 \cdot 10^{-04}$	(−92.2)	14
ED_{444}	$1.16851 \cdot 10^{-03}$	(−91.1)	$1.64835 \cdot 10^{-04}$	(−92.1)	15
ED_{445}	$1.31972 \cdot 10^{-03}$	(−90.0)	$1.64835 \cdot 10^{-04}$	(−92.1)	16
ED_{555}	$4.29224 \cdot 10^{-03}$	(−67.4)	$1.73120 \cdot 10^{-04}$	(−91.7)	18
ED_{777}	$1.08119 \cdot 10^{-02}$	(−17.8)	$2.96304 \cdot 10^{-04}$	(−85.8)	24

Table 3 Comparison of various advanced zig–zag theories to evaluate the transverse displacement amplitude (center plate deflection) $\hat{u}_z = u_z \frac{100E_{\text{core}}}{zP^t h(\frac{a}{h})^4}$ in $z = z_{\text{bottom}}^{\text{upper skin}} = \frac{3}{10}h, x = a/2, y = b/2$

a/h	4		100		
		FCSR = 10^1			
Elasticity	3.01123	Err.%	1.51021	Err.%	DOF
EDZ_{111}	2.34412	(−22.2)	1.15866	(−23.3)	9
EDZ_{115}	2.68809	(−10.7)	1.50975	(−0.03)	13
EDZ_{225}	2.97597	(−1.17)	1.51015	(0.00)	15
EDZ_{335}	2.98229	(−0.96)	1.51017	(0.00)	17
EDZ_{444}	2.97886	(−1.07)	1.51017	(0.00)	18
EDZ_{445}	2.98242	(−0.96)	1.51017	(0.00)	19
EDZ_{555}	2.98737	(−0.79)	1.51018	(0.00)	21
EDZ_{777}	2.99670	(−0.48)	1.51019	(0.00)	27
		FCSR = 10^5			
Elasticity	$1.31593 \cdot 10^{-02}$	Error %	$2.08948 \cdot 10^{-03}$	Error %	
EDZ_{111}	$8.36735 \cdot 10^{-04}$	(−93.6)	$1.63329 \cdot 10^{-04}$	(−92.2)	9
EDZ_{115}	$7.93264 \cdot 10^{-03}$	(−39.7)	$1.64323 \cdot 10^{-04}$	(−92.1)	13
EDZ_{225}	$9.52709 \cdot 10^{-03}$	(−27.6)	$2.24726 \cdot 10^{-04}$	(−89.2)	15
EDZ_{335}	$1.02216 \cdot 10^{-02}$	(−22.3)	$2.70584 \cdot 10^{-04}$	(−87.0)	17
EDZ_{444}	$1.26288 \cdot 10^{-02}$	(−4.03)	$1.16305 \cdot 10^{-03}$	(−44.3)	18
EDZ_{445}	$1.26702 \cdot 10^{-02}$	(−3.72)	$1.16345 \cdot 10^{-03}$	(−44.3)	19
EDZ_{555}	$1.30409 \cdot 10^{-02}$	(−0.90)	$1.78411 \cdot 10^{-03}$	(−14.6)	21
EDZ_{777}	$1.31363 \cdot 10^{-02}$	(−0.17)	$2.02060 \cdot 10^{-03}$	(−3.30)	27

Table 4 Comparison of various layerwise theories to evaluate the transverse shear stress $\hat{\sigma}_{zx} = \frac{\sigma_{zx}}{zP^t(\frac{a}{h})}$ in $z = z_{\text{bottom}}^{\text{upper skin}} = \frac{3}{10}h$, $x = 0, y = b/2$. The indefinite equilibrium equations have been integrated along the thickness

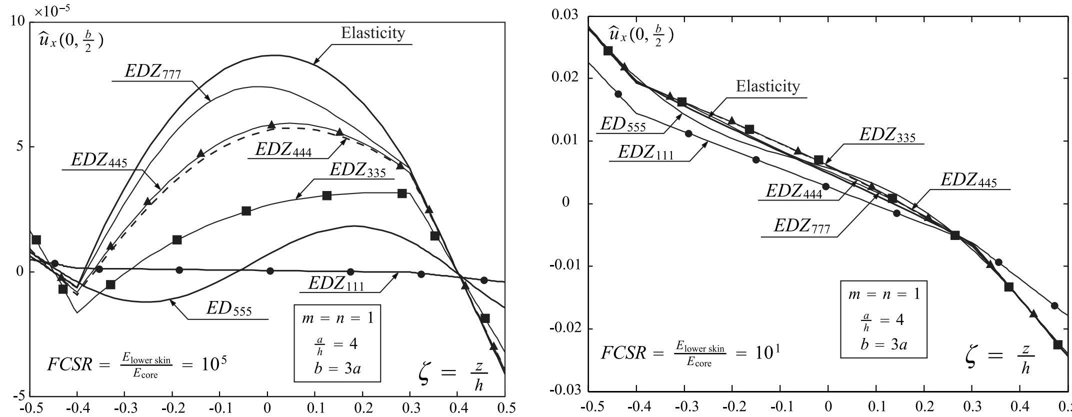
a/h	4	Err.	100	Error	
		FCSR = 10^1			
Elasticity	0.32168	Error %	0.33176	Error %	DOF
LD_{111}	0.31730	(−1.36)	0.32345	(−2.50)	12
LD_{115}	0.32157	(−0.03)	0.33177	(0.00)	24
LD_{225}	0.32142	(−0.08)	0.33176	(0.00)	30
LD_{555}	0.32168	(0.00)	0.33176	(0.00)	48
		FCSR = 10^5			
Elasticity	$5.40842 \cdot 10^{-04}$	Error %	0.27797	Error %	
LD_{111}	$1.05700 \cdot 10^{-04}$	(−80.5)	0.26143	(−5.95)	12
LD_{115}	$3.25506 \cdot 10^{-04}$	(−39.8)	0.27797	(0.00)	24
LD_{225}	$5.40033 \cdot 10^{-04}$	(−0.15)	0.27797	(0.00)	30
LD_{555}	$5.40842 \cdot 10^{-04}$	(0.00)	0.27797	(0.00)	48

Table 5 Comparison of various AHSDT to evaluate the transverse shear stress $\hat{\sigma}_{zx} = \frac{\sigma_{zx}}{zP^t(\frac{a}{h})}$ in $z = z_{\text{bottom}}^{\text{upper skin}} = \frac{3}{10}h, x = 0$, $y = b/2$. The indefinite equilibrium equations have been integrated along the thickness

a/h	4	Error	100	Error	
		FCSR = 10^1			
Elasticity	0.32168	Error %	0.33176	Error %	DOF
ED_{111}	0.33178	(+3.14)	0.33178	(+0.01)	6
ED_{115}	0.32509	(+1.06)	0.33177	(0.00)	10
ED_{225}	0.32503	(+1.04)	0.33177	(0.00)	12
ED_{335}	0.32502	(+1.04)	0.33177	(0.00)	14
ED_{444}	0.33240	(+3.33)	0.33178	(+0.01)	15
ED_{445}	0.32976	(+2.51)	0.33178	(+0.01)	16
ED_{555}	0.32884	(+2.23)	0.33178	(+0.01)	18
ED_{777}	0.32707	(+1.68)	0.33177	(0.00)	24
		FCSR = 10^5			
Elasticity	$5.40842 \cdot 10^{-04}$	Error %	0.27797	Error %	
ED_{111}	0.33242	(>100)	0.33242	(+19.6)	6
ED_{115}	0.33277	(>100)	0.33243	(+19.6)	10
ED_{225}	0.33166	(>100)	0.33243	(+19.6)	12
ED_{335}	0.32018	(>100)	0.33241	(+19.6)	14
ED_{444}	0.30529	(>100)	0.33238	(+16.6)	15
ED_{445}	0.30566	(>100)	0.33238	(+16.6)	16
ED_{555}	0.21639	(>100)	0.33214	(+19.5)	18
ED_{777}	$3.96907 \cdot 10^{-02}$	(>100)	0.32865	(+18.2)	24

Table 6 Comparison of various advanced zig–zag theories to evaluate the transverse shear stress $\hat{\sigma}_{zx} = \frac{\sigma_{zx}}{zP^t(\frac{a}{h})}$ in $z = z_{\text{bottom}}^{\text{upper skin}} = \frac{3}{10}h, x = 0, y = b/2$. The indefinite equilibrium equations have been integrated along the thickness

a/h	4	Error	100	Error	
	FCSR = 10^1				
Elasticity	0.32168	Error %	0.33176	Error %	DOF
EDZ_{111}	0.34184	(+6.27)	0.34497	(+3.98)	9
EDZ_{115}	0.32854	(+2.13)	0.33178	(+0.01)	13
EDZ_{225}	0.32856	(+2.14)	0.33178	(+0.01)	15
EDZ_{335}	0.32803	(+1.97)	0.33178	(+0.01)	17
EDZ_{444}	0.32913	(+2.32)	0.33178	(+0.01)	18
EDZ_{445}	0.32783	(+1.91)	0.33178	(+0.01)	19
EDZ_{555}	0.32755	(+1.82)	0.33177	(0.00)	21
EDZ_{777}	0.32530	(+1.12)	0.33177	(+0.00)	27
		FCSR = 10^5			
Elasticity	$5.40842 \cdot 10^{-04}$	Err.%	0.27797	Err.%	
EDZ_{111}	0.30971	(>100)	0.33077	(+19.0)	9
EDZ_{115}	$8.48754 \cdot 10^{-02}$	(>100)	0.33240	(+19.6)	13
EDZ_{225}	$5.88764 \cdot 10^{-02}$	(>100)	0.33069	(+19.0)	15
EDZ_{335}	$4.74947 \cdot 10^{-02}$	(>100)	0.32939	(+18.5)	17
EDZ_{444}	$6.84336 \cdot 10^{-03}$	(>100)	0.30392	(+9.34)	18
EDZ_{445}	$6.84955 \cdot 10^{-03}$	(>100)	0.30405	(+9.38)	19
EDZ_{555}	$1.87520 \cdot 10^{-03}$	(>100)	0.28655	(+3.09)	21
EDZ_{777}	$8.02443 \cdot 10^{-04}$	(+48.4)	0.27994	(+0.71)	27



a) b)

Fig. 12 Dimensionless in-plane displacement $\hat{u}_x$.

All the results have been compared with the solution obtained by solving the exact problem [13]. The exact value is indicated with the terminology *elasticity* and is the reference value corresponding to the solution of the differential equations that govern the problem. The details of this elasticity solution are here omitted for brevity [13].

Tables 1–6 compare a large number of ALWT, AHSDT, and AZZT. The orders of the expansions along the thickness are changed. FCSR is dramatically changed to clearly show the limitations of some of the possible approaches. From Tables 1–6 it is deduced that when FCSR is very high most of the AHSDTs fail to predict the correct displacements or stresses. For example, when $\text{FCSR} = 10^5$ even the AHSDT with order 5 (or larger) of all the displacements has a large error [see Table 2 and Fig. 12a)]. The zig–zag theories present a better compromise [see Fig. 12a)] but only the layerwise theories can achieve excellent accuracy even for this very challenging case (thick plate with extremely high FCSR). This is demonstrated in Tables 1 and 4. When FCSR is not very high the equivalent single layer theories present a very good approximation. In particular, the zig–zag theories are capable to capture the correct displacement and stress fields even for thick plates (see Fig. 12b). This is due to the fact that the zig–zag form of the displacements is enforced a priori by the means of Murakami's zig–zag function. The GUF allows to compare different types of theories and different types of expansions without any effort. It can be then considered a natural tool to study the numerical performances of multilayered structures in general (see [58]). The sensitivity with respect to the type of theory and/or orders of expansion is an impractical task with the classical approaches but this does not present problems when the GUF is used. A virtually infinite number of layerwise, zig–zag, and higher-order theories can be generated with any effort.

Conclusions

An invariant methodology, the GUF, to include practically any type of axiomatic theory has been presented. ∞^3 higher-order shear deformation theories, ∞^3 zig–zag theories, and ∞^3 layerwise theories, with any combination of orders of expansion for the displacement variables can be derived from six invariant 1×1 matrices. This property gives the possibility to include in a single software and FEM formulation classical and advanced models. It is then possible to find the best formulation, among the infinite possible choices, for the problem under investigation. The applicability of the GUF is not confined to the displacement-based case. Mixed variational statements can in fact be used and multifield loadings could be included as well. In any case all the theories are derived from 1×1 invariant matrices (the so-called kernels of the GUF). The number of independent kernels depend on the variational statement that is used. In the case of displacement-based theories only six kernels are required, whereas in the case of mixed variational statements this number is different.

The possibility to select the type of theory among the infinite possibilities without actually change the software makes the GUF particularly indicated for the solution of optimization and probabilistic problems and for the quasi-3D analysis of structures when localized effects are important. The GUF can also have an important educational feature because it allows the user to adopt practically all the existing approaches and to compare with an infinite number of more advanced models. The engineer has only to learn a single invariant formulation and software and this can be applied to study a very large variety of new cases. When the CPU time is more important then a low order theory can be selected and when the accuracy is the determinant factor the order can be increased or the theory can be changed for a more advanced analysis.

The GUF is also a powerful tool to evaluate the sensitivity with respect the order of expansion or the type of theory (e.g., higher-order shear deformation theories with or without zig–zag effects included in the model). This property makes possible to tailor the software to the peculiarity of a new problem with unknown requirements in terms of axiomatic theories necessary to study it. An adaptive FEM software can be designed using the GUF: the software can change the orders or the theory without the actual need of a new code or theoretical development.

The paper presented an application of these concepts for a sandwich structure. It is demonstrated that when the face-to-core stiffness ratio is very high the classical approaches fail and a layerwise formulation is required. This finding is immediate with the GUF because all the possible combinations of orders are allowed and also different types of theories can be experimented as well. It is concluded that the GUF is a modern approach for the theoretical and computational analysis of multilayered structures.

Appendix A

This appendix presents the explicit expressions for the in-plane and out-of-plane contributions of the virtual work ([see the left-hand side of Eq. (10)]

$$\begin{aligned}
\delta\boldsymbol{\varepsilon}_{pG}^{kT}\boldsymbol{\sigma}_{pH}^{k} &= \delta\varepsilon_{xxG}^{k}\tilde{C}_{11}^{k}\varepsilon_{xxG}^{k} + \delta\varepsilon_{xxG}^{k}\tilde{C}_{12}^{k}\varepsilon_{yyG}^{k} + \delta\varepsilon_{xxG}^{k}\tilde{C}_{16}^{k}\gamma_{xyG}^{k} \\
&+ \delta\varepsilon_{xxG}^{k}\tilde{C}_{13}^{k}\varepsilon_{zzG}^{k} + \delta\varepsilon_{yyG}^{k}\tilde{C}_{12}^{k}\varepsilon_{xxG}^{k} + \delta\varepsilon_{yyG}^{k}\tilde{C}_{22}^{k}\varepsilon_{yyG}^{k} \\
&+ \delta\varepsilon_{yyG}^{k}\tilde{C}_{26}^{k}\gamma_{xyG}^{k} + \delta\varepsilon_{yyG}^{k}\tilde{C}_{23}^{k}\varepsilon_{zzG}^{k} + \delta\gamma_{xyG}^{k}\tilde{C}_{16}^{k}\varepsilon_{xxG}^{k} \\
&+ \delta\gamma_{xyG}^{k}\tilde{C}_{26}^{k}\varepsilon_{yyG}^{k} + \delta\gamma_{xyG}^{k}\tilde{C}_{66}^{k}\gamma_{xyG}^{k} + \delta\gamma_{xyG}^{k}\tilde{C}_{36}^{k}\varepsilon_{zzG}^{k}
\end{aligned} \tag{A1}$$

Using the geometric relations [Eq. (11)]

$$
\begin{aligned}
\delta\boldsymbol{\varepsilon}_{pG}^{kT}\boldsymbol{\sigma}_{pH}^{k} = {} & \tilde{C}_{11}^{k}\,{}^{x}F_{\alpha_{u_x}}\,{}^{x}F_{\beta_{u_x}}\,\delta u^{k}_{x\alpha_{u_x},x}\,u^{k}_{x\beta_{u_x},x} \\
& + \tilde{C}_{12}^{k}\,{}^{x}F_{\alpha_{u_x}}\,{}^{y}F_{\beta_{u_y}}\,\delta u^{k}_{x\alpha_{u_x},x}\,u^{k}_{y\beta_{u_y},y} + \tilde{C}_{16}^{k}\,{}^{x}F_{\alpha_{u_x}}\,{}^{x}F_{\beta_{u_x}}\,\delta u^{k}_{x\alpha_{u_x},x}\,u^{k}_{x\beta_{u_x},y} \\
& + \tilde{C}_{16}^{k}\,{}^{x}F_{\alpha_{u_x}}\,{}^{y}F_{\beta_{u_y}}\,\delta u^{k}_{x\alpha_{u_x},x}\,u^{k}_{y\beta_{u_y},x} + \tilde{C}_{13}^{k}\,{}^{x}F_{\alpha_{u_x}}\,{}^{z}F_{\beta_{u_z},z}\,\delta u^{k}_{x\alpha_{u_x},x}\,u^{k}_{z\beta_{u_z}} \\
& + \tilde{C}_{12}^{k}\,{}^{y}F_{\alpha_{u_y}}\,{}^{x}F_{\beta_{u_x}}\,\delta u^{k}_{y\alpha_{u_y},y}\,u^{k}_{x\beta_{u_x},x} + \tilde{C}_{22}^{k}\,{}^{y}F_{\alpha_{u_y}}\,{}^{y}F_{\beta_{u_y}}\,\delta u^{k}_{y\alpha_{u_y},y}\,u^{k}_{y\beta_{u_y},y} \\
& + \tilde{C}_{26}^{k}\,{}^{y}F_{\alpha_{u_y}}\,{}^{x}F_{\beta_{u_x}}\,\delta u^{k}_{y\alpha_{u_y},y}\,u^{k}_{x\beta_{u_x},y} + \tilde{C}_{26}^{k}\,{}^{y}F_{\alpha_{u_y}}\,{}^{y}F_{\beta_{u_y}}\,\delta u^{k}_{y\alpha_{u_y},y}\,u^{k}_{y\beta_{u_y},x} \\
& + \tilde{C}_{23}^{k}\,{}^{y}F_{\alpha_{u_y}}\,{}^{z}F_{\beta_{u_z},z}\,\delta u^{k}_{y\alpha_{u_y},y}\,u^{k}_{z\beta_{u_z}} + \tilde{C}_{16}^{k}\,{}^{x}F_{\alpha_{u_x}}\,{}^{x}F_{\beta_{u_x}}\,\delta u^{k}_{x\alpha_{u_x},y}\,u^{k}_{x\beta_{u_x},x} \\
& + \tilde{C}_{16}^{k}\,{}^{y}F_{\alpha_{u_y}}\,{}^{x}F_{\beta_{u_x}}\,\delta u^{k}_{y\alpha_{u_y},x}\,u^{k}_{x\beta_{u_x},x} + \tilde{C}_{26}^{k}\,{}^{x}F_{\alpha_{u_x}}\,{}^{y}F_{\beta_{u_y}}\,\delta u^{k}_{x\alpha_{u_x},y}\,u^{k}_{y\beta_{u_y},y} \\
& + \tilde{C}_{26}^{k}\,{}^{y}F_{\alpha_{u_y}}\,{}^{y}F_{\beta_{u_y}}\,\delta u^{k}_{y\alpha_{u_y},x}\,u^{k}_{y\beta_{u_y},y} + \tilde{C}_{66}^{k}\,{}^{x}F_{\alpha_{u_x}}\,{}^{x}F_{\beta_{u_x}}\,\delta u^{k}_{x\alpha_{u_x},y}\,u^{k}_{x\beta_{u_x},y} \\
& + \tilde{C}_{66}^{k}\,{}^{y}F_{\alpha_{u_y}}\,{}^{x}F_{\beta_{u_x}}\,\delta u^{k}_{y\alpha_{u_y},x}\,u^{k}_{x\beta_{u_x},y} + \tilde{C}_{66}^{k}\,{}^{x}F_{\alpha_{u_x}}\,{}^{y}F_{\beta_{u_y}}\,\delta u^{k}_{x\alpha_{u_x},y}\,u^{k}_{y\beta_{u_y},x} \\
& + \tilde{C}_{66}^{k}\,{}^{y}F_{\alpha_{u_y}}\,{}^{y}F_{\beta_{u_y}}\,\delta u^{k}_{y\alpha_{u_y},x}\,u^{k}_{y\beta_{u_y},x} + \tilde{C}_{36}^{k}\,{}^{x}F_{\alpha_{u_x}}\,{}^{z}F_{\beta_{u_z},z}\,\delta u^{k}_{x\alpha_{u_x},y}\,u^{k}_{z\beta_{u_z}} \\
& + \tilde{C}_{36}^{k}\,{}^{y}F_{\alpha_{u_y}}\,{}^{z}F_{\beta_{u_z},z}\,\delta u^{k}_{y\alpha_{u_y},x}\,u^{k}_{z\beta_{u_z}} \qquad \text{(A2)}
\end{aligned}
$$

Now focus is on the function $\delta\boldsymbol{\varepsilon}_{nG}^{kT}\boldsymbol{\sigma}_{nH}^{k}$

$$
\begin{aligned}
\delta\boldsymbol{\varepsilon}_{nG}^{kT}\boldsymbol{\sigma}_{nH}^{k} = {} & \delta\gamma^{k}_{xzG}\tilde{C}_{55}^{k}\gamma^{k}_{xzG} + \delta\gamma^{k}_{xzG}\tilde{C}_{45}^{k}\gamma^{k}_{yzG} + \delta\gamma^{k}_{yzG}\tilde{C}_{45}^{k}\gamma^{k}_{xzG} \\
& + \delta\gamma^{k}_{yzG}\tilde{C}_{44}^{k}\gamma^{k}_{yzG} + \delta\varepsilon^{k}_{zz}\tilde{C}_{13}^{k}\varepsilon^{k}_{xxG} + \delta\varepsilon^{k}_{zzG}\tilde{C}_{23}^{k}\varepsilon^{k}_{yyG} \\
& + \delta\varepsilon^{k}_{zzG}\tilde{C}_{36}^{k}\gamma^{k}_{xyG} + \delta\varepsilon^{k}_{zzG}\tilde{C}_{33}^{k}\varepsilon^{k}_{zzG} \qquad \text{(A3)}
\end{aligned}
$$

Using the geometric relations [Eq. (11)]

$$
\begin{aligned}
\delta\boldsymbol{\varepsilon}_{nG}^{kT}\boldsymbol{\sigma}_{nH}^{k} = {} & \tilde{C}_{55}^{k}\,{}^{z}F_{\alpha_{u_z}}\,{}^{z}F_{\beta_{u_z}}\,\delta u^{k}_{z\alpha_{u_z},x}\,u^{k}_{z\beta_{u_z},x} \\
& + \tilde{C}_{55}^{k}\,{}^{x}F_{\alpha_{u_x},z}\,{}^{z}F_{\beta_{u_z}}\,\delta u^{k}_{x\alpha_{u_x}}\,u^{k}_{z\beta_{u_z},x} + \tilde{C}_{55}^{k}\,{}^{z}F_{\alpha_{u_z}}\,{}^{x}F_{\beta_{u_x},z}\,\delta u^{k}_{z\alpha_{u_z},x}\,u^{k}_{x\beta_{u_x}} \\
& + \tilde{C}_{55}^{k}\,{}^{x}F_{\alpha_{u_x},z}\,{}^{x}F_{\beta_{u_x},z}\,\delta u^{k}_{x\alpha_{u_x}}\,u^{k}_{x\beta_{u_x}} + \tilde{C}_{45}^{k}\,{}^{z}F_{\alpha_{u_z}}\,{}^{z}F_{\beta_{u_z}}\,\delta u^{k}_{z\alpha_{u_z},x}\,u^{k}_{z\beta_{u_z},y} \\
& + \tilde{C}_{45}^{k}\,{}^{x}F_{\alpha_{u_x},z}\,{}^{z}F_{\beta_{u_z}}\,\delta u^{k}_{x\alpha_{u_x}}\,u^{k}_{z\beta_{u_z},y} + \tilde{C}_{45}^{k}\,{}^{z}F_{\alpha_{u_z}}\,{}^{y}F_{\beta_{u_y},z}\,\delta u^{k}_{z\alpha_{u_z},x}\,u^{k}_{y\beta_{u_y}} \\
& + \tilde{C}_{45}^{k}\,{}^{x}F_{\alpha_{u_x},z}\,{}^{y}F_{\beta_{u_y},z}\,\delta u^{k}_{x\alpha_{u_x}}\,u^{k}_{y\beta_{u_y}} + \tilde{C}_{45}^{k}\,{}^{z}F_{\alpha_{u_z}}\,{}^{z}F_{\beta_{u_z}}\,\delta u^{k}_{z\alpha_{u_z},y}\,u^{k}_{z\beta_{u_z},x} \\
& + \tilde{C}_{45}^{k}\,{}^{y}F_{\alpha_{u_y},z}\,{}^{z}F_{\beta_{u_z}}\,\delta u^{k}_{y\alpha_{u_y}}\,u^{k}_{z\beta_{u_z},x} + \tilde{C}_{45}^{k}\,{}^{z}F_{\alpha_{u_z}}\,{}^{x}F_{\beta_{u_x},z}\,\delta u^{k}_{z\alpha_{u_z},y}\,u^{k}_{x\beta_{u_x}} \\
& + \tilde{C}_{45}^{k}\,{}^{y}F_{\alpha_{u_y},z}\,{}^{x}F_{\beta_{u_x},z}\,\delta u^{k}_{y\alpha_{u_y}}\,u^{k}_{x\beta_{u_x}} + \tilde{C}_{44}^{k}\,{}^{z}F_{\alpha_{u_z}}\,{}^{z}F_{\beta_{u_z}}\,\delta u^{k}_{z\alpha_{u_z},y}\,u^{k}_{z\beta_{u_z},y} \\
& + \tilde{C}_{44}^{k}\,{}^{y}F_{\alpha_{u_y},z}\,{}^{z}F_{\beta_{u_z}}\,\delta u^{k}_{y\alpha_{u_y}}\,u^{k}_{z\beta_{u_z},y} + \tilde{C}_{44}^{k}\,{}^{z}F_{\alpha_{u_z}}\,{}^{y}F_{\beta_{u_y},z}\,\delta u^{k}_{z\alpha_{u_z},y}\,u^{k}_{y\beta_{u_y}} \\
& + \tilde{C}_{44}^{k}\,{}^{y}F_{\alpha_{u_y},z}\,{}^{y}F_{\beta_{u_y},z}\,\delta u^{k}_{y\alpha_{u_y}}\,u^{k}_{y\beta_{u_y}} + \tilde{C}_{13}^{k}\,{}^{z}F_{\alpha_{u_z},z}\,{}^{x}F_{\beta_{u_x}}\,\delta u^{k}_{z\alpha_{u_z}}\,u^{k}_{x\beta_{u_x},x} \\
& + \tilde{C}_{23}^{k}\,{}^{z}F_{\alpha_{u_z},z}\,{}^{y}F_{\beta_{u_y}}\,\delta u^{k}_{z\alpha_{u_z}}\,u^{k}_{y\beta_{u_y},y} + \tilde{C}_{36}^{k}\,{}^{z}F_{\alpha_{u_z},z}\,{}^{x}F_{\beta_{u_x}}\,\delta u^{k}_{z\alpha_{u_z}}\,u^{k}_{x\beta_{u_x},y} \\
& + \tilde{C}_{36}^{k}\,{}^{z}F_{\alpha_{u_z},z}\,{}^{y}F_{\beta_{u_y}}\,\delta u^{k}_{z\alpha_{u_z}}\,u^{k}_{y\beta_{u_y},x} + \tilde{C}_{33}^{k}\,{}^{z}F_{\alpha_{u_z},z}\,{}^{z}F_{\beta_{u_z},z}\,\delta u^{k}_{z\alpha_{u_z}}\,u^{k}_{z\beta_{u_z}} \qquad \text{(A4)}
\end{aligned}
$$

Appendix B

This appendix presents the explicit expressions for virtual external work [see Eqs. (23) and (24)]

$$
\begin{aligned}
\delta L_e^k = {} & + \int_{\Omega^k} \delta u_{x\alpha_{u_x}}\,{}^{t}D^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\,P^{kt}_{x\beta_{u_x}}\,\mathrm{d}x\,\mathrm{d}y \\
& + \int_{\Omega^k} \delta u_{y\alpha_{u_y}}\,{}^{t}D^{k\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\,P^{kt}_{y\beta_{u_y}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega^k} \delta u_{z\alpha_{u_z}}\,{}^{t}D^{k\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\,P^{kt}_{z\beta_{u_z}}\,\mathrm{d}x\,\mathrm{d}y \\
& + \int_{\Omega^k} \delta u_{x\alpha_{u_x}}\,{}^{b}D^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\,P^{kb}_{x\beta_{u_x}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega^k} \delta u_{y\alpha_{u_y}}\,{}^{b}D^{k\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\,P^{kb}_{y\beta_{u_y}}\,\mathrm{d}x\,\mathrm{d}y \\
& + \int_{\Omega^k} \delta u_{z\alpha_{u_z}}\,{}^{b}D^{k\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\,P^{kb}_{z\beta_{u_z}}\,\mathrm{d}x\,\mathrm{d}y \\
& + \int_{\Gamma_\sigma^k}\int_{z_{\mathrm{bot}_k}}^{z_{\mathrm{top}_k}} [\delta u_n^k \bar{\sigma}^k_{nn} + \delta u_s^k \bar{\sigma}^k_{ns} + \delta u_z^k \bar{\sigma}^k_{nz}]\,\mathrm{d}z\,\mathrm{d}s \qquad \text{(B1)}
\end{aligned}
$$

After a few elaborations it can be demonstrated that the contribution of layer k to the expression of virtual external work is

$$
\begin{aligned}
\delta L_e^k = {} & + \int_{\Omega^k} \delta u^k_{x\alpha_{u_x}}\,{}^{t}D^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\,P^{kt}_{x\beta_{u_x}}\,\mathrm{d}x\,\mathrm{d}y \\
& + \int_{\Omega^k} \delta u^k_{y\alpha_{u_y}}\,{}^{t}D^{k\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\,P^{kt}_{y\beta_{u_y}}\,\mathrm{d}x\,\mathrm{d}y \\
& + \int_{\Omega^k} \delta u^k_{z\alpha_{u_z}}\,{}^{t}D^{k\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\,P^{kt}_{z\beta_{u_z}}\,\mathrm{d}x\,\mathrm{d}y \\
& + \int_{\Omega^k} \delta u^k_{x\alpha_{u_x}}\,{}^{b}D^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\,P^{kb}_{x\beta_{u_x}}\,\mathrm{d}x\,\mathrm{d}y \\
& + \int_{\Omega^k} \delta u^k_{y\alpha_{u_y}}\,{}^{b}D^{k\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\,P^{kb}_{y\beta_{u_y}}\,\mathrm{d}x\,\mathrm{d}y \\
& + \int_{\Omega^k} \delta u^k_{z\alpha_{u_z}}\,{}^{b}D^{k\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\,P^{kb}_{z\beta_{u_z}}\,\mathrm{d}x\,\mathrm{d}y \\
& + Z^{k\alpha_{u_x}\beta_{u_x}}_{u_x u_x} \int_{\Gamma_\sigma^{kx}} \delta u^k_{x\alpha_{u_x}}\,\bar{t}^k_{x\beta_{u_x}}\,\mathrm{d}s \\
& + Z^{k\alpha_{u_y}\beta_{u_y}}_{u_y u_y} \int_{\Gamma_\sigma^{ky}} \delta u^k_{y\alpha_{u_y}}\,\bar{t}^k_{y\beta_{u_y}}\,\mathrm{d}s \\
& + Z^{k\alpha_{u_z}\beta_{u_z}}_{u_z u_z} \int_{\Gamma_\sigma^{kz}} \delta u^k_{z\alpha_{u_z}}\,\bar{t}^k_{z\beta_{u_z}}\,\mathrm{d}s \qquad \text{(B2)}
\end{aligned}
$$

Appendix C

This appendix presents the explicit expressions for the governing equations [see Eqs. (26–28)]

$$
\begin{aligned}
\delta u^k_{x\alpha_{u_x}}:\; & -Z^{k\alpha_{u_x}\beta_{u_x}}_{11u_x u_x}\,u^k_{x\beta_{u_x},x,x} - Z^{k\alpha_{u_x}\beta_{u_y}}_{12u_x u_y}\,u^k_{y\beta_{u_y},y,x} - Z^{k\alpha_{u_x}\beta_{u_x}}_{16u_x u_x}\,u^k_{x\beta_{u_x},y,x} \\
& - Z^{k\alpha_{u_x}\beta_{u_y}}_{16u_x u_y}\,u^k_{y\beta_{u_y},x,x} - Z^{k\alpha_{u_x}\beta_{u_z,z}}_{13u_x u_z}\,u^k_{z\beta_{u_z},x} - Z^{k\alpha_{u_x}\beta_{u_x}}_{16u_x u_x}\,u^k_{x\beta_{u_x},x,y} \\
& - Z^{k\alpha_{u_x}\beta_{u_y}}_{26u_x u_y}\,u^k_{y\beta_{u_y},y,y} - Z^{k\alpha_{u_x}\beta_{u_x}}_{66u_x u_x}\,u^k_{x\beta_{u_x},y,y} - Z^{k\alpha_{u_x}\beta_{u_y}}_{66u_x u_y}\,u^k_{y\beta_{u_y},x,y} \\
& - Z^{k\alpha_{u_x}\beta_{u_z,z}}_{36u_x u_z}\,u^k_{z\beta_{u_z},y} + Z^{k\alpha_{u_x,z}\beta_{u_z}}_{55u_x u_z}\,u^k_{z\beta_{u_z},x} + Z^{k\alpha_{u_x,z}\beta_{u_x,z}}_{55u_x u_x}\,u^k_{x\beta_{u_x}} \\
& + Z^{k\alpha_{u_x,z}\beta_{u_y,z}}_{45u_x u_y}\,u^k_{y\beta_{u_y}} + Z^{k\alpha_{u_x,z}\beta_{u_z}}_{45u_x u_z}\,u^k_{z\beta_{u_z},y} - D^{kt\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\,P^{kt}_{x\beta_{u_x}} \\
& - D^{kb\alpha_{u_x}\beta_{u_x}}_{u_x u_x}\,P^{kb}_{x\beta_{u_x}} = 0 \qquad \text{(C1)}
\end{aligned}
$$

$$
\begin{aligned}
\delta u^k_{y\alpha_{u_y}}:\; & -Z^{k\alpha_{u_y}\beta_{u_x}}_{12u_y u_x}\,u^k_{x\beta_{u_x},x,y} - Z^{k\alpha_{u_y}\beta_{u_y}}_{22u_y u_y}\,u^k_{y\beta_{u_y},y,y} - Z^{k\alpha_{u_y}\beta_{u_x}}_{26u_y u_x}\,u^k_{x\beta_{u_x},y,y} \\
& - Z^{k\alpha_{u_y}\beta_{u_y}}_{26u_y u_y}\,u^k_{y\beta_{u_y},x,y} - Z^{k\alpha_{u_y}\beta_{u_z,z}}_{23u_y u_z}\,u^k_{z\beta_{u_z},y} - Z^{k\alpha_{u_y}\beta_{u_x}}_{16u_y u_x}\,u^k_{x\beta_{u_x},x,x} \\
& - Z^{k\alpha_{u_y}\beta_{u_y}}_{26u_y u_y}\,u^k_{y\beta_{u_y},y,x} - Z^{k\alpha_{u_y}\beta_{u_x}}_{66u_y u_x}\,u^k_{x\beta_{u_x},y,x} - Z^{k\alpha_{u_y}\beta_{u_y}}_{66u_y u_y}\,u^k_{y\beta_{u_y},x,x} \\
& - Z^{k\alpha_{u_y}\beta_{u_z,z}}_{36u_y u_z}\,u^k_{z\beta_{u_z},x} + Z^{k\alpha_{u_y,z}\beta_{u_z}}_{45u_y u_z}\,u^k_{z\beta_{u_z},x} + Z^{k\alpha_{u_y,z}\beta_{u_x,z}}_{45u_y u_x}\,u^k_{x\beta_{u_x}} \\
& + Z^{k\alpha_{u_y,z}\beta_{u_z}}_{44u_y u_z}\,u^k_{z\beta_{u_z},y} + Z^{k\alpha_{u_y,z}\beta_{u_y,z}}_{44u_y u_y}\,u^k_{y\beta_{u_y}} - D^{kt\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\,P^{kt}_{y\beta_{u_y}} \\
& - D^{kb\alpha_{u_y}\beta_{u_y}}_{u_y u_y}\,P^{kb}_{y\beta_{u_y}} = 0 \qquad \text{(C2)}
\end{aligned}
$$

$$
\begin{aligned}
\delta u^k_{z\alpha_{u_z}}:\; & -Z^{k\alpha_{u_z}\beta_{u_z}}_{55u_z u_z}\,u^k_{z\beta_{u_z},x,x} - Z^{k\alpha_{u_z}\beta_{u_x,z}}_{55u_z u_x}\,u^k_{x\beta_{u_x},x} - Z^{k\alpha_{u_z}\beta_{u_z}}_{45u_z u_z}\,u^k_{z\beta_{u_z},y,x} \\
& - Z^{k\alpha_{u_z}\beta_{u_y,z}}_{45u_z u_y}\,u^k_{y\beta_{u_y},x} - Z^{k\alpha_{u_z}\beta_{u_z}}_{45u_z u_z}\,u^k_{z\beta_{u_z},x,y} - Z^{k\alpha_{u_z}\beta_{u_x,z}}_{45u_z u_x}\,u^k_{x\beta_{u_x},y} \\
& - Z^{k\alpha_{u_z}\beta_{u_z}}_{44u_z u_z}\,u^k_{z\beta_{u_z},y,y} - Z^{k\alpha_{u_z}\beta_{u_y,z}}_{44u_z u_y}\,u^k_{y\beta_{u_y},y} + Z^{k\alpha_{u_z,z}\beta_{u_x}}_{13u_z u_x}\,u^k_{x\beta_{u_x},x} \\
& + Z^{k\alpha_{u_z,z}\beta_{u_y}}_{23u_z u_y}\,u^k_{y\beta_{u_y},y} + Z^{k\alpha_{u_z,z}\beta_{u_x}}_{36u_z u_x}\,u^k_{x\beta_{u_x},y} + Z^{k\alpha_{u_z,z}\beta_{u_y}}_{36u_z u_y}\,u^k_{y\beta_{u_y},x} \\
& + Z^{k\alpha_{u_z,z}\beta_{u_z,z}}_{33u_z u_z}\,u^k_{z\beta_{u_z}} - D^{kt\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\,P^{kt}_{z\beta_{u_z}} - D^{kb\alpha_{u_z}\beta_{u_z}}_{u_z u_z}\,P^{kb}_{z\beta_{u_z}} = 0 \qquad \text{(C3)}
\end{aligned}
$$

Appendix D

This appendix presents the explicit expressions for the invariant kernels of the GUF [see Eq. (34)]. The expressions for the kernels follow

$$
\begin{aligned}
K_{u_x u_x}^{k\alpha_{u_x}\beta_{u_x}} &= +Z_{11u_x u_x}^{k\alpha_{u_x}\beta_{u_x}}\frac{m^2\pi^2}{a^2} + Z_{66u_x u_x}^{k\alpha_{u_x}\beta_{u_x}}\frac{n^2\pi^2}{b^2} + Z_{55u_x u_x}^{k\alpha_{u_{x,z}}\beta_{u_{x,z}}}\\
K_{u_x u_y}^{k\alpha_{u_x}\beta_{u_y}} &= +Z_{12u_x u_y}^{k\alpha_{u_x}\beta_{u_y}}\frac{mn\pi^2}{ab} + Z_{66u_x u_y}^{k\alpha_{u_x}\beta_{u_y}}\frac{mn\pi^2}{ab}\\
K_{u_x u_z}^{k\alpha_{u_x}\beta_{u_z}} &= -Z_{13u_x u_z}^{k\alpha_{u_x}\beta_{u_{z,z}}}\frac{m\pi}{a} + Z_{55u_x u_z}^{k\alpha_{u_{x,z}}\beta_{u_z}}\frac{m\pi}{a}\\
K_{u_y u_x}^{k\alpha_{u_y}\beta_{u_x}} &= +Z_{12u_y u_x}^{k\alpha_{u_y}\beta_{u_x}}\frac{mn\pi^2}{ab} + Z_{66u_y u_x}^{k\alpha_{u_y}\beta_{u_x}}\frac{mn\pi^2}{ab}\\
K_{u_y u_y}^{k\alpha_{u_y}\beta_{u_y}} &= +Z_{22u_y u_y}^{k\alpha_{u_y}\beta_{u_y}}\frac{n^2\pi^2}{b^2} + Z_{66u_y u_y}^{k\alpha_{u_y}\beta_{u_y}}\frac{m^2\pi^2}{a^2} + Z_{44u_y u_y}^{k\alpha_{u_{y,z}}\beta_{u_{y,z}}}\\
K_{u_y u_z}^{k\alpha_{u_y}\beta_{u_z}} &= -Z_{23u_y u_z}^{k\alpha_{u_y}\beta_{u_{z,z}}}\frac{n\pi}{b} + Z_{44u_y u_z}^{k\alpha_{u_{y,z}}\beta_{u_z}}\frac{n\pi}{b}\\
K_{u_z u_x}^{k\alpha_{u_z}\beta_{u_x}} &= +Z_{55u_z u_x}^{k\alpha_{u_z}\beta_{u_{x,z}}}\frac{m\pi}{a} - Z_{13u_z u_x}^{k\alpha_{u_{z,z}}\beta_{u_x}}\frac{m\pi}{a}\\
K_{u_z u_y}^{k\alpha_{u_z}\beta_{u_y}} &= +Z_{44u_z u_y}^{k\alpha_{u_z}\beta_{u_{y,z}}}\frac{n\pi}{b} - Z_{23u_z u_y}^{k\alpha_{u_{z,z}}\beta_{u_y}}\frac{n\pi}{b}\\
K_{u_z u_z}^{k\alpha_{u_z}\beta_{u_z}} &= +Z_{55u_z u_z}^{k\alpha_{u_z}\beta_{u_z}}\frac{m^2\pi^2}{a^2} + Z_{44u_z u_z}^{k\alpha_{u_z}\beta_{u_z}}\frac{n^2\pi^2}{b^2} + Z_{33u_z u_z}^{k\alpha_{u_{z,z}}\beta_{u_{z,z}}}
\end{aligned}
\tag{D1}
$$

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F. Pai
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